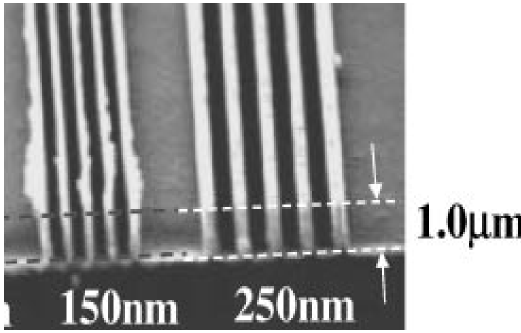
A Novel Finite Element Formulation for Large Deformation Analysis Based on Incremental Equilibrium Equation in Conjunction with Rezoning Technique

Yuki ONISHI, Kenji AMAYA

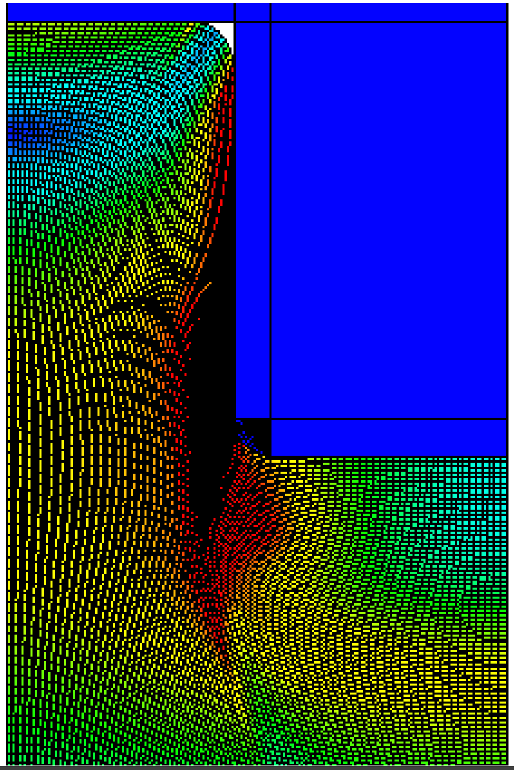
Tokyo Institute of Technology (Japan)

Motivation and Background

- We want to solve **severely large deformation** problems **accurately and stably**!
  (Final target: thermal nanoimprinting)



- Finite elements are **distorted** in a short time, thereby resulting in convergence failure.



FE rezoning method (*h*-adaptive, mesh-to-mesh solution mapping) is indispensable.

Methods for Forming Simulation

	Software	Accuracy	Stability
One Step Method	HyperForm FASTFORM	★	★★★★
Dynamic-Explicit FE Rezoning	LS-DYNA PAM-STAMP	★★	★★★
Static-Explicit FE Rezoning	ASU/P-form	★★★	★★
Static-Implicit FE Rezoning	ABAQUS MARC	★★★★	★

Most of the rezoning researches try to improve this.

Our approach tries to improve this.

But How ?

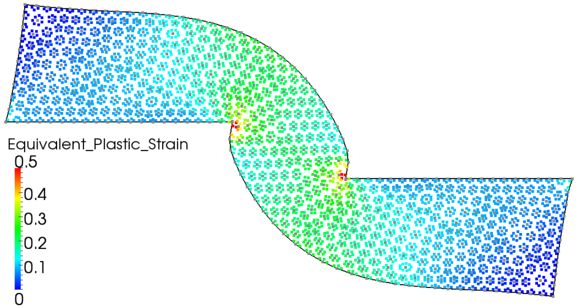
Idea of Our Approach

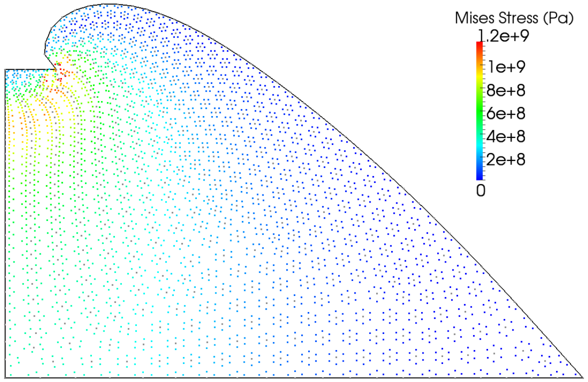
Our Idea

We adopts
the incremental equilibrium equation (IEE)
as the equation to be solved.

The IEE is recently developed in a meshfree method.

See the e-book of PARTICLES2011 or our full-paper of IJNME in press.





Objective

Develop an **accurate and stable**
implicit FE rezoning method
for large deformation problems
based on
the incremental equilibrium equation (IEE)

Table of Body Contents

① Derivation of the IEE for static-implicit analysis
② Formulation of our implicit FE rezoning method based on the IEE
③ Examples of analysis

①
Derivation of the incremental equilibrium equation (IEE) for static-implicit analysis

Virtual Work Equation in Rate Form

$$\int_{\Omega(t)} \dot{\mathbf{\Pi}}_t^T(t) : \delta \boldsymbol{F}_t(t)\ \mathrm{d}\Omega = \int_{\Gamma(t)} \dot{\underline{\boldsymbol{t}}}_t(t) \cdot \delta \boldsymbol{u}\ \mathrm{d}\Gamma + \int_{\Omega(t)} \rho \dot{\boldsymbol{g}} \cdot \delta \boldsymbol{u}\ \mathrm{d}\Omega$$

Work Conjugate

$\square_t$: Variable in the Current Configuration,

$\delta\square$: Variation, $\dot{\square}$: Material Time Derivative,

$\mathbf{\Pi}$: 1st Piola-Kirchhoff Stress Tensor,

$\boldsymbol{F}$: Deformation Gradient Tensor,

$\underline{\boldsymbol{t}}$: Surface Traction Vector,

Ω: Analysis Domain, Γ: Domain Boundary,

$\boldsymbol{u}$: Displacement vector, ρ: Density,

$\boldsymbol{g}$: Body Force Vector

Linearization and Discretization

$$\int_{\Omega(t)} \dot{\mathbf{\Pi}}_t^T(t) : \delta \boldsymbol{F}_t(t)\ \mathrm{d}\Omega$$

Work Conjugate

$$= \int_{\Gamma(t)} \dot{\underline{\boldsymbol{t}}}_t(t) \cdot \delta \boldsymbol{u}\ \mathrm{d}\Gamma + \int_{\Omega(t)} \rho \dot{\boldsymbol{g}} \cdot \delta \boldsymbol{u}\ \mathrm{d}\Omega$$

Linearization in Time

$$\dot{\mathbf{\Pi}}_t^T(t) \simeq \Delta \mathbf{\Pi}_t^T / \Delta t, \quad \dot{\underline{\boldsymbol{t}}}_t(t) \simeq \Delta \underline{\boldsymbol{t}}_t / \Delta t, \quad \dot{\boldsymbol{g}} \simeq \Delta \boldsymbol{g} / \Delta t$$

Galerkin Discretization

$$\delta \boldsymbol{F}_t(t) \simeq [B_{\mathrm{N}}]\{\delta u\}, \quad \delta \boldsymbol{u} \simeq \{N\}\{\delta u\}$$

fully implicit time advancing

$$\sum_{e \in \mathbb{E}} \int_{\Omega_e^+} [B_{\mathrm{N}}^+]^T \{\Delta \Pi_t^T\}\ \mathrm{d}\Omega$$

$$= \sum_{s \in \mathbb{S}} \int_{\Gamma_s^+} [N^+]^T \{\Delta \underline{t}_t\}\ \mathrm{d}\Gamma + \sum_{e \in \mathbb{E}} \int_{\Omega_e^+} \rho^+ [N^+]^T \{\Delta g\}\ \mathrm{d}\Omega$$

"+" : Trial Variable,
E : Set of Elements,
S : Set of Element Faces

Incremental Equilibrium Equation (IEE)

$$\sum_{e\in\mathbb{E}} \int_{\Omega_e^+} [B_{\mathrm{N}}^+]^T \{\Delta\Pi_t^T\} \, \mathrm{d}\Omega = \sum_{s\in\mathbb{S}} \int_{\Gamma_s^+} [N^+]^T \{\Delta \underline{t}_t\} \, \mathrm{d}\Gamma + \sum_{e\in\mathbb{E}} \int_{\Omega_e^+} \rho^+ [N^+]^T \{\Delta g\} \, \mathrm{d}\Omega$$

"+" : Trial Variable,
E : Set of Elements,
S : Set of Element Faces

Let the left-hand side be $\{\Delta f^{\mathrm{int}}\}$,
the right-hand side be $\{\Delta f^{\mathrm{ext}}\}$.

The IEE: $$\{\Delta f^{\mathrm{ext}}\} - \{\Delta f^{\mathrm{int}}\} = \{0\}$$

Avoid error accumulation through timesteps.

The IEE (secondary form):

$$(\{f^{\mathrm{ext}}\} + \{\Delta f^{\mathrm{ext}}\}) - (\{f^{\mathrm{int}}\} + \{\Delta f^{\mathrm{int}}\}) = \{0\}$$

We use the secondary form in the actual implementation.

Comparison of IEE to Standard EE

[Standard EE]

$$\{f^{\mathrm{ext}}\} - \{f^{\mathrm{int}}\} = \{0\},$$

$$\{f^{\mathrm{ext}}\} = \sum_{s\in\mathbb{S}} \int_{\Gamma_s^+} [N^+]^T \{\underline{t}^+\}\ \mathrm{d}\Gamma + \sum_{e\in\mathbb{E}} \int_{\Omega_e^+} \rho^+ [N^+]^T \{g\}\ \mathrm{d}\Omega,$$

$$\{f^{\mathrm{int}}\} = \sum_{e\in\mathbb{E}} \int_{\Omega_e^+} [B_{\mathrm{L}}^+]^T \{T^+\}\ \mathrm{d}\Omega,$$

[IEE]

$$\{\Delta f^{\mathrm{ext}}\} - \{\Delta f^{\mathrm{int}}\} = \{0\},$$

$$\{\Delta f^{\mathrm{ext}}\} = \sum_{s\in\mathbb{S}} \int_{\Gamma_s^+} [N^+]^T \{\Delta \underline{t}_t\}\ \mathrm{d}\Gamma + \sum_{e\in\mathbb{E}} \int_{\Omega_e^+} \rho^+ [N^+]^T \{\Delta g\}\ \mathrm{d}\Omega,$$

$$\{\Delta f^{\mathrm{int}}\} = \sum_{e\in\mathbb{E}} \int_{\Omega_e^+} [B_{\mathrm{N}}^+]^T \{\Delta \Pi_t^T\}\ \mathrm{d}\Omega,$$

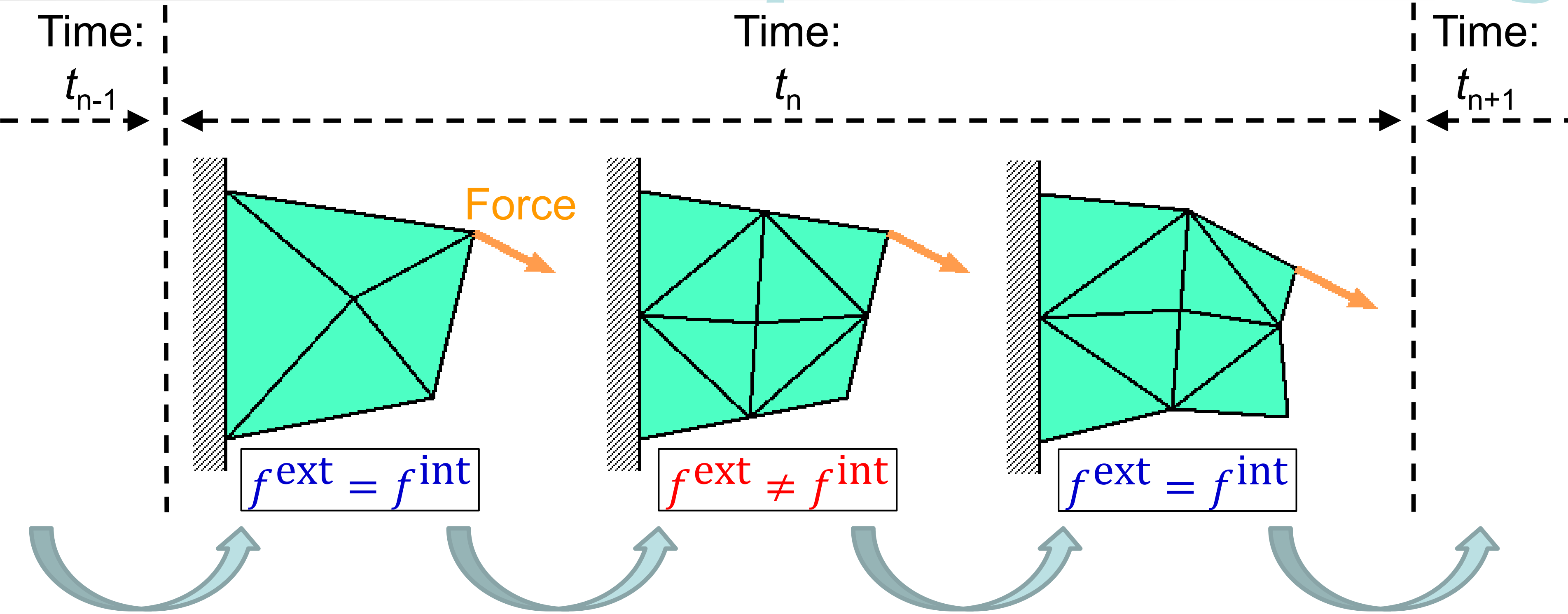
②

Formulation of

our implicit FE rezoning method

based on the IEE

Conventional Implicit FE Rezoning



Solve
Standard EE

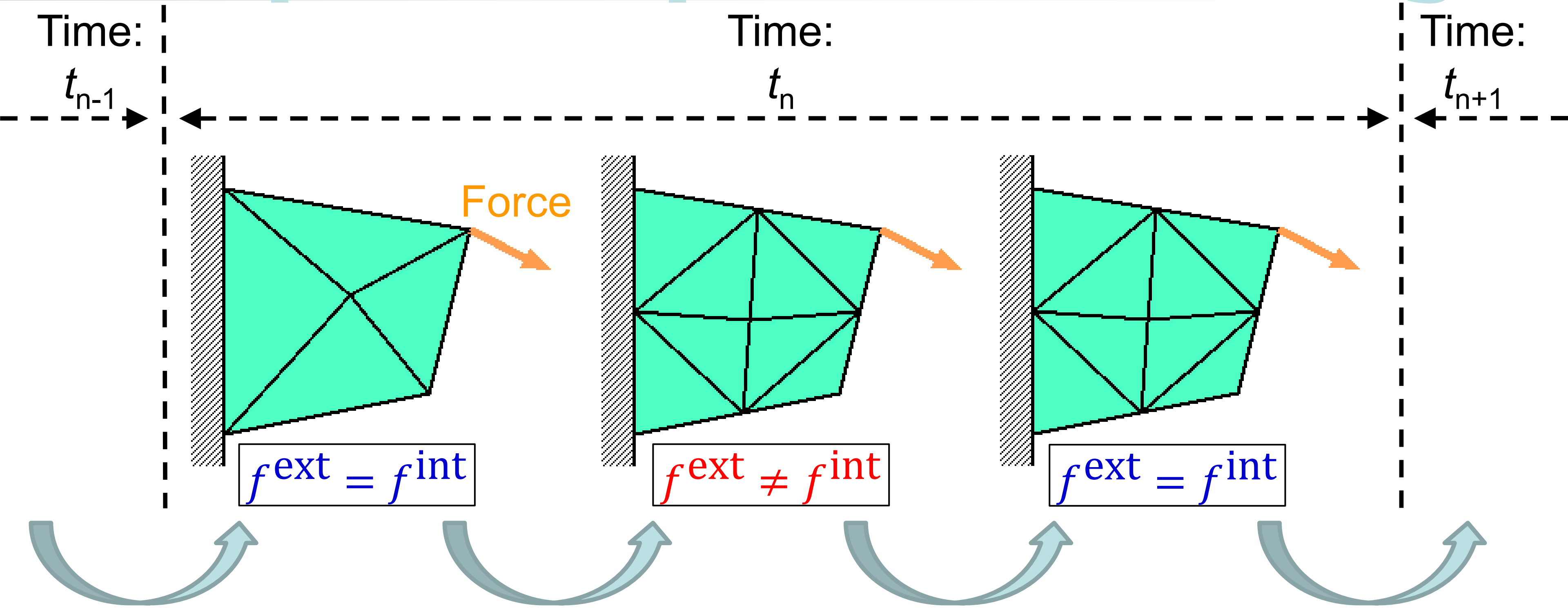
Remesh
and
Map States

Resolve
Standard EE

Solve
Standard EE

Induce
Large Deformation
and
Convergence Failure

Proposed Implicit FE Rezoning



Solve IEE

Remesh and Map States including f^{ext}

Assign f^{ext} to f^{int}

Solve IEE

Unique Operation

Never Induce Any Deformation and Convergence Failure

Flowchart of the Proposed Method

- Start of timestep loop
  - Assume initial $\{\Delta u\}$
  - Start of Newton-Raphson loop
    - Calculate trial states
    - Calculate $\{\Delta f^{\mathrm{ext}}\}, \{\Delta f^{\mathrm{int}}\}$, and $[K]$
    - Convergence check
    - Solve $[K]\{\delta u\} = (\{f^{\mathrm{ext}}\} + \{\Delta f^{\mathrm{ext}}\}) - (\{f^{\mathrm{int}}\} + \{\Delta f^{\mathrm{int}}\})$
    - Substitute $\{\Delta u\} + \{\delta u\}$ for $\{\Delta u\}$
  - Substitute $\{f^{\mathrm{ext}}\} + \{\Delta f^{\mathrm{ext}}\}$ for $\{f^{\mathrm{ext}}\}$
  - Substitute $\{f^{\mathrm{int}}\} + \{\Delta f^{\mathrm{int}}\}$ for $\{f^{\mathrm{int}}\}$
  - Update States
  - Rezone if necessary

Almost the same as the conventional method except the green parts

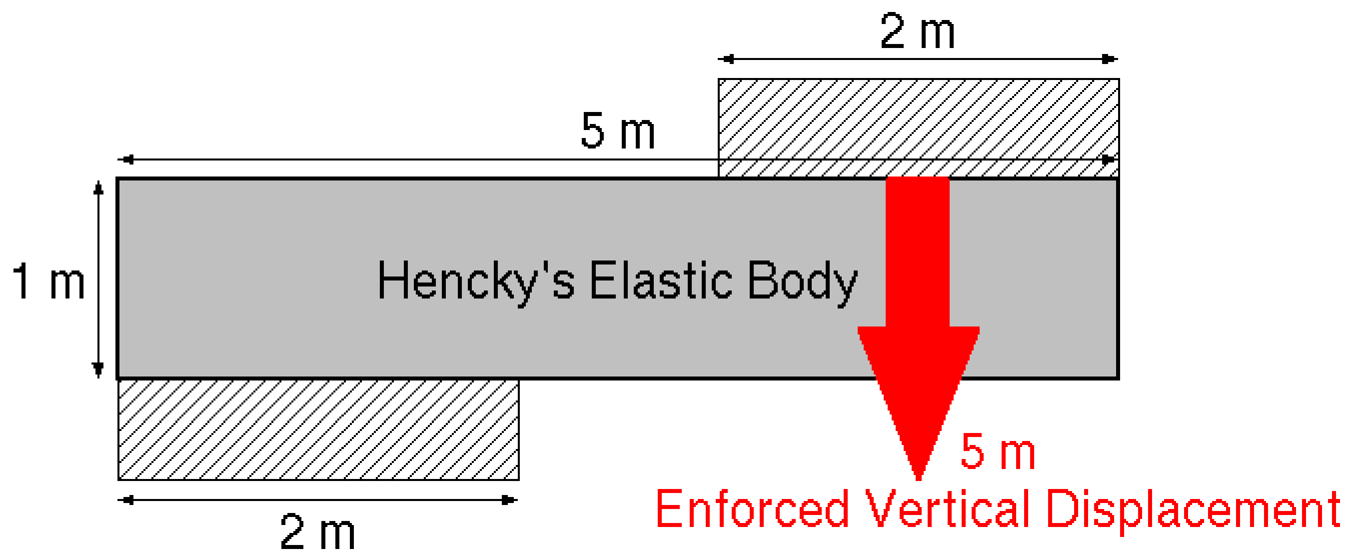
Proposed vs. Conventional

	Proposed Implicit FE Rezoning	Conventional Implicit FE Rezoning
Equation to be Solved	IEE	Standard EE
Mapping of f^{ext}	Required…	Unnecessary!
Equilibrium after Mapping	YES!	NO…
Unique Deformed Shape at a Time	YES!	NO…
Convergence Failure in Rezoning Process	NO!	YES…

③

Examples of analysis

Shearing of 2D Bar



- Static, 2D Plane-strain condition
- All 1st order triangular elements
- Global rezoning every 10 timesteps (much more frequent than necessary)
- remeshing with ANSYS GAMBIT

Shearing of 2D Bar

- Material: Hencky's elastic body
  - constitutive equation in total strain form:

$$\boldsymbol{T} = \boldsymbol{C}_{\mathrm{L}} : \boldsymbol{E}$$

Cauchy Stress ∝ Hencky Strain

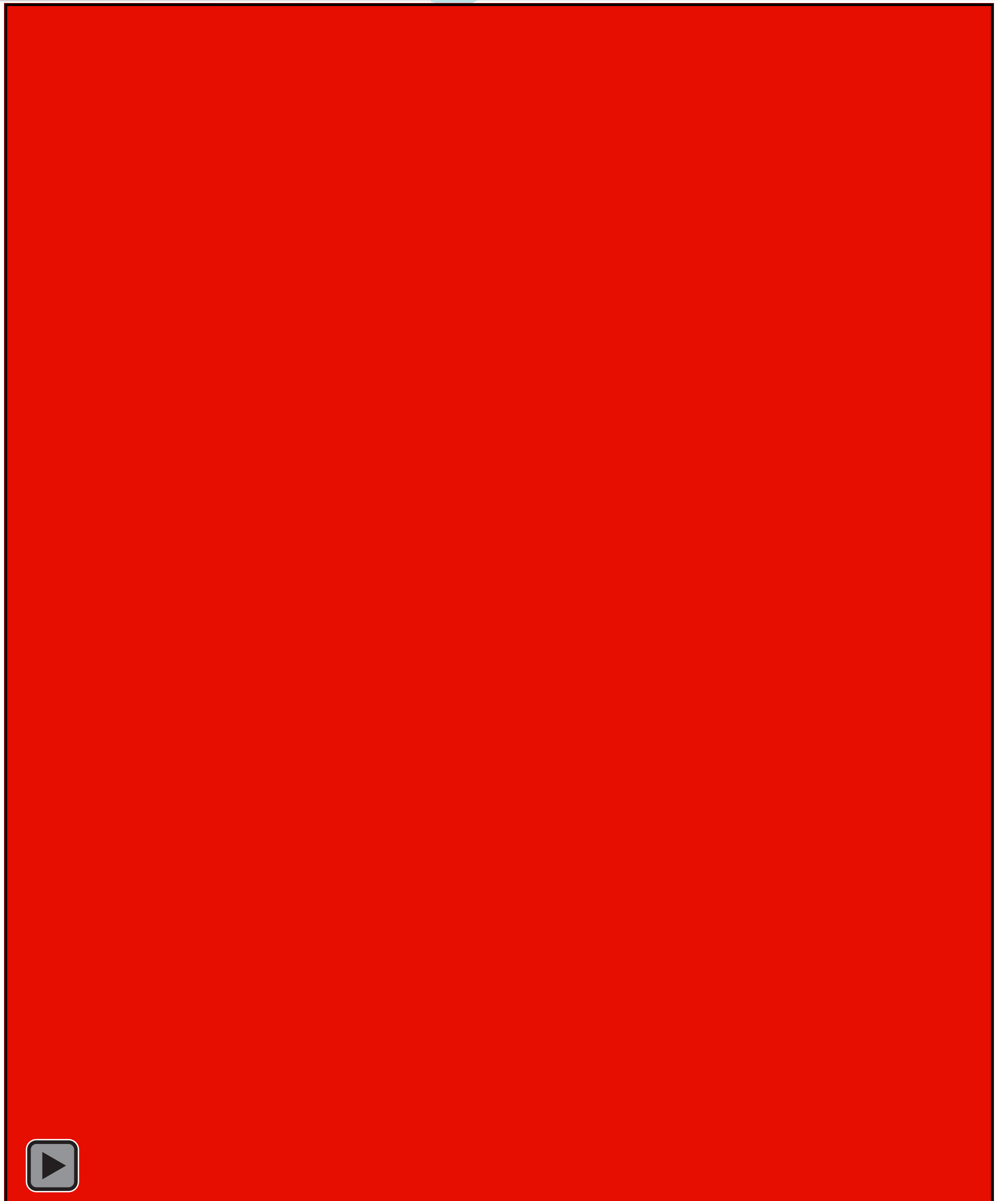
  - constitutive equation in rate form:

$$\boxed{\overset{\circ}{\boldsymbol{T}} = \boldsymbol{C}_{\mathrm{L}} : \boldsymbol{D}}$$

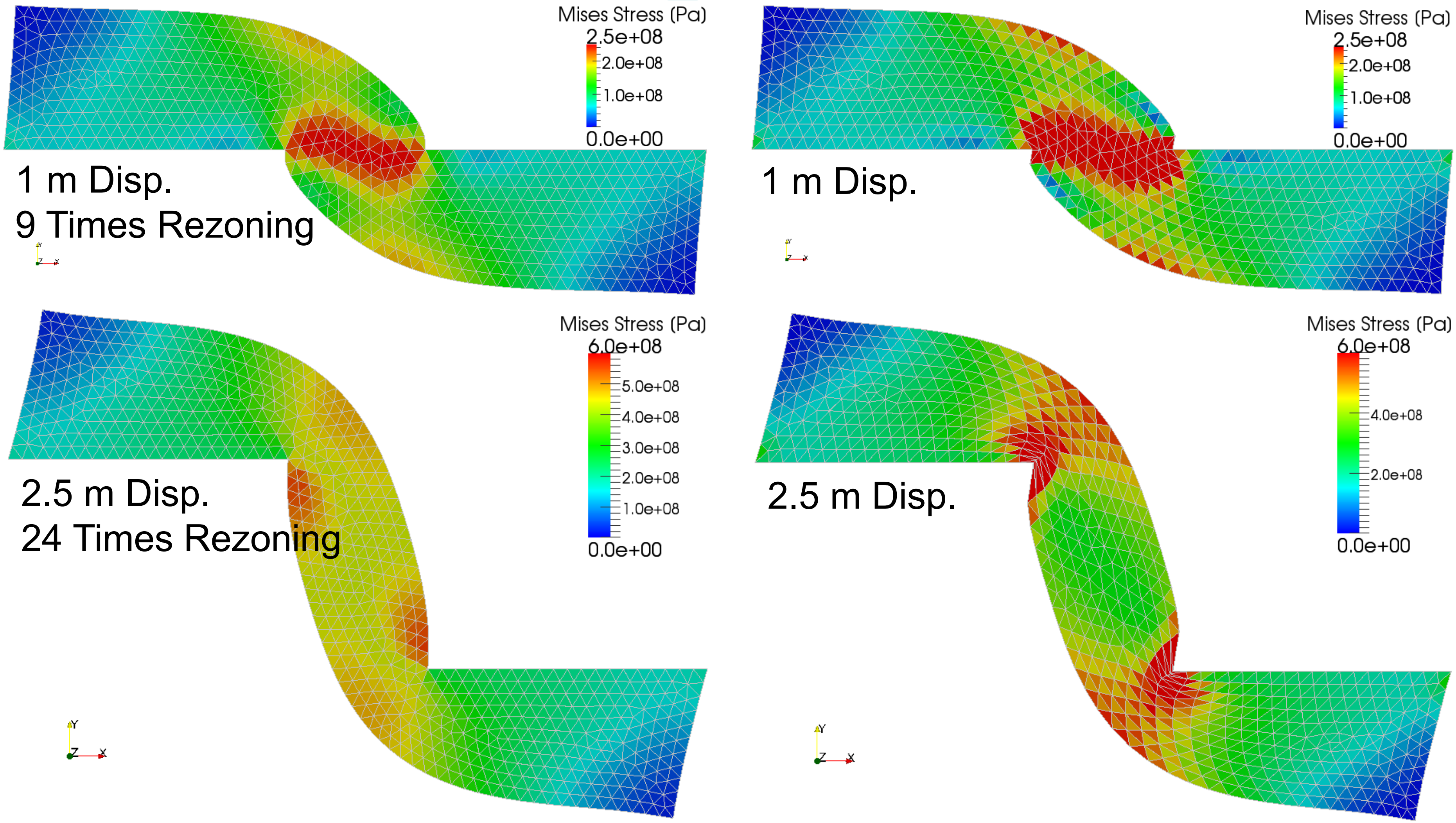
Jaumann Rate of Cauchy Stress ∝ Stretching

  - Young's modulus: 1 GPa; Poisson's Ratio: 0.3

Shearing of 2D Bar



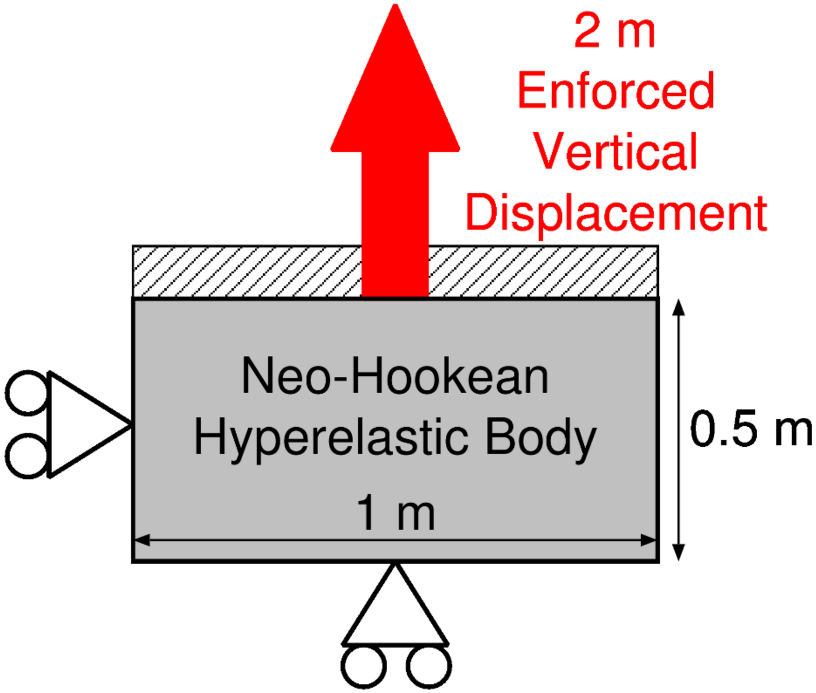
Shearing of 2D Bar



Proposed Method

Standard FEM without Rezoning

Tension of 2D Brick



- Static, 2D Plane-strain condition
- All 1st order triangular elements
- Global rezoning every 5 timesteps

Tension of 2D Brick

■ Material: Neo-Hookean hyperelastic body

- Strain energy density function:

$$U = C_{10}(\bar{I}_1 - 3) + \frac{1}{D_1}(J-1)^2$$

- Constitutive equation in total strain form:

$$\boldsymbol{T} = \frac{2}{J} C_{10}\,\mathrm{dev}(\bar{\boldsymbol{B}}) + \frac{2}{D_1}(J-1)\boldsymbol{I}$$

- Constitutive equation in strain rate form:

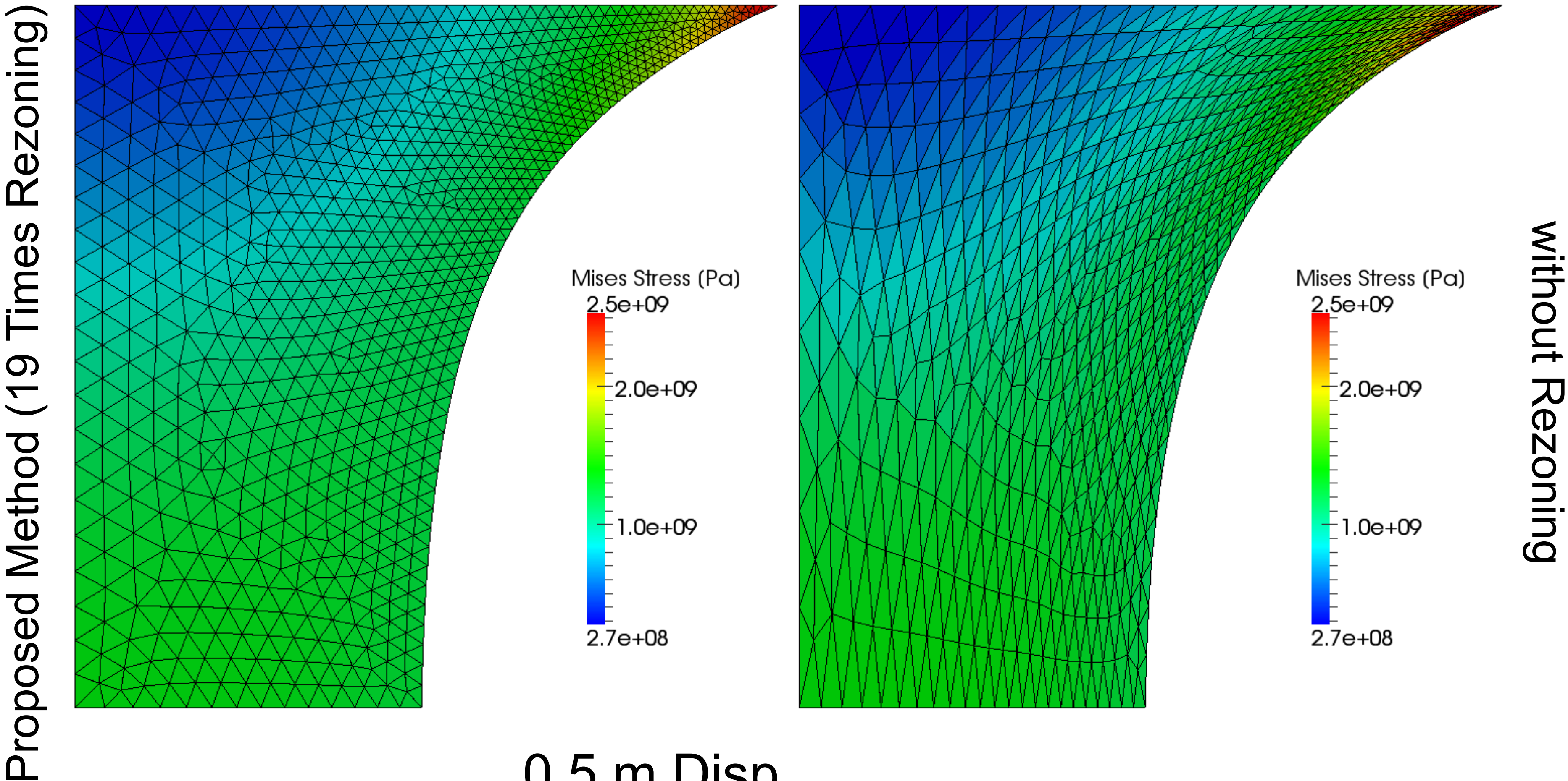
$$\overset{\circ}{\boldsymbol{T}} = \boldsymbol{C}_{\mathrm{L}}(\boldsymbol{F}) : \boldsymbol{D}$$

where $C_L(F)$ is obtained through a long hand calculation.

- C_{10}=0.172 GPa; D_1=0.6 GPa^{-1}

Tension of 2D Brick

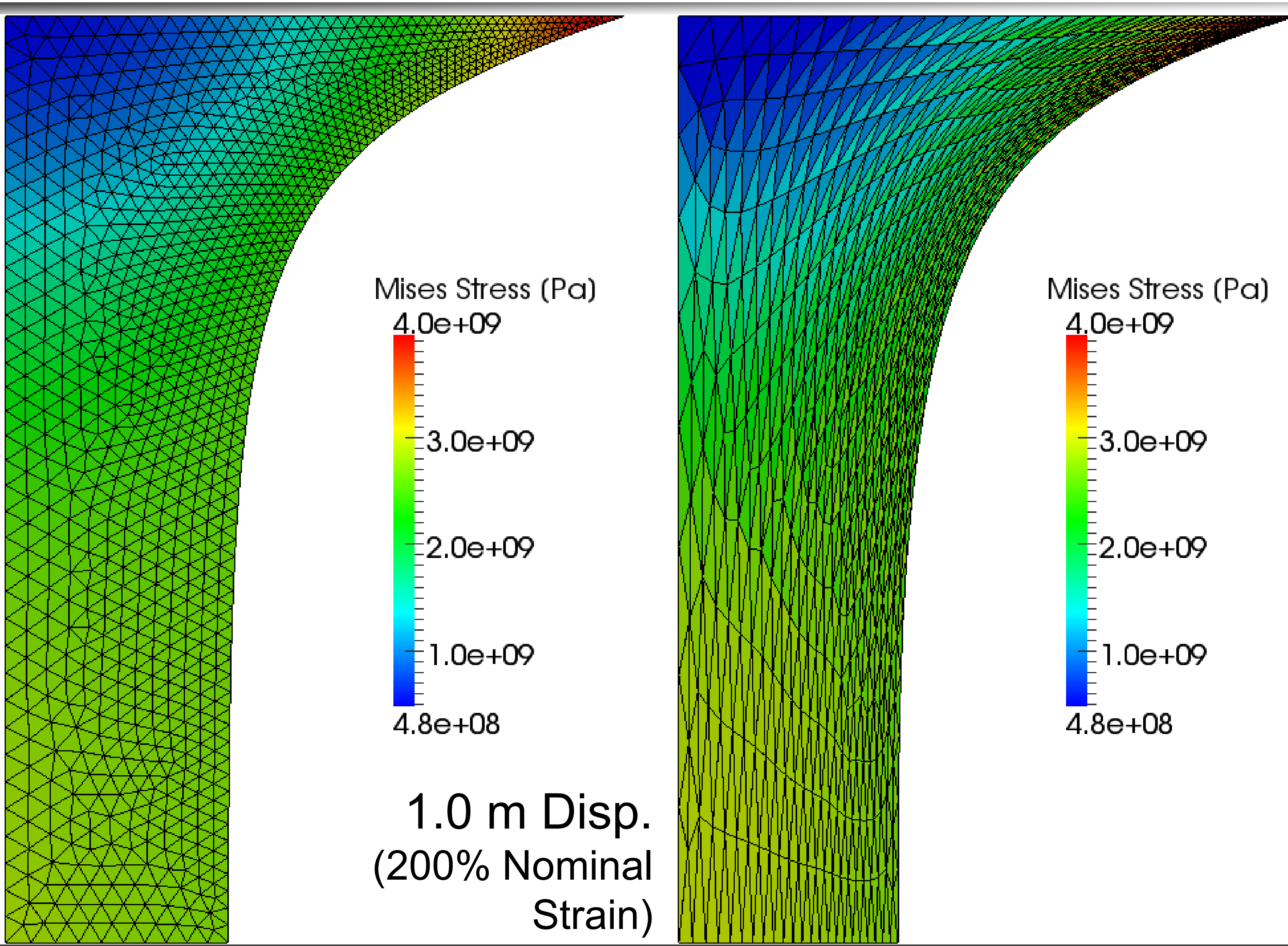
Tension of 2D Brick



0.5 m Disp.
(100% Nominal Strain)

Tension of 2D Brick

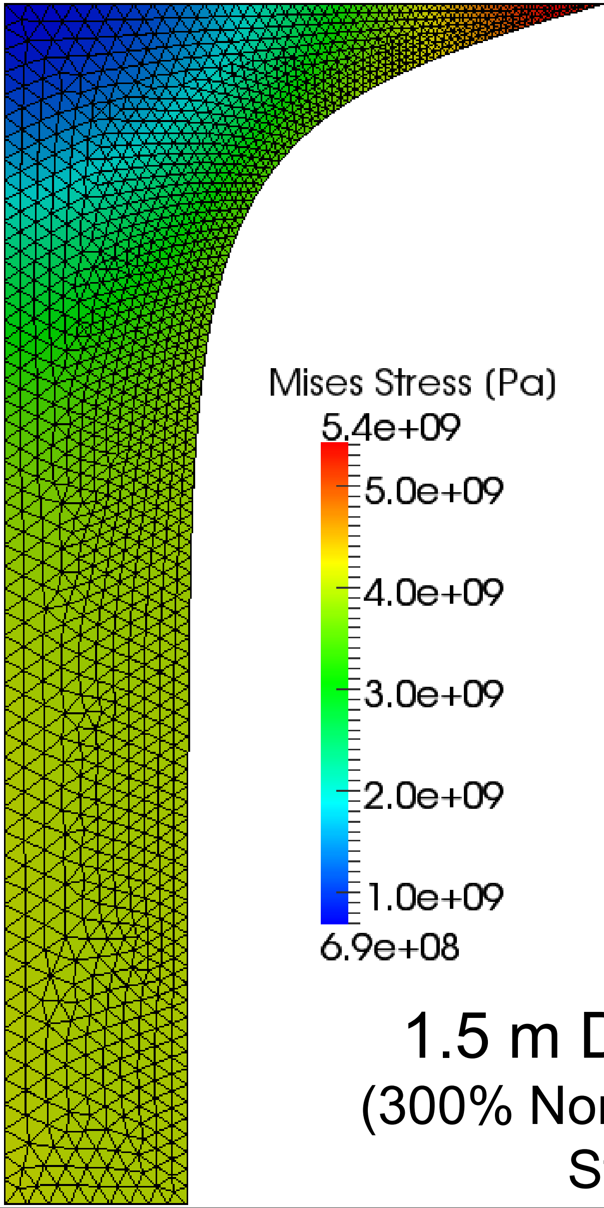
Proposed Method (39 Times Rezoning)



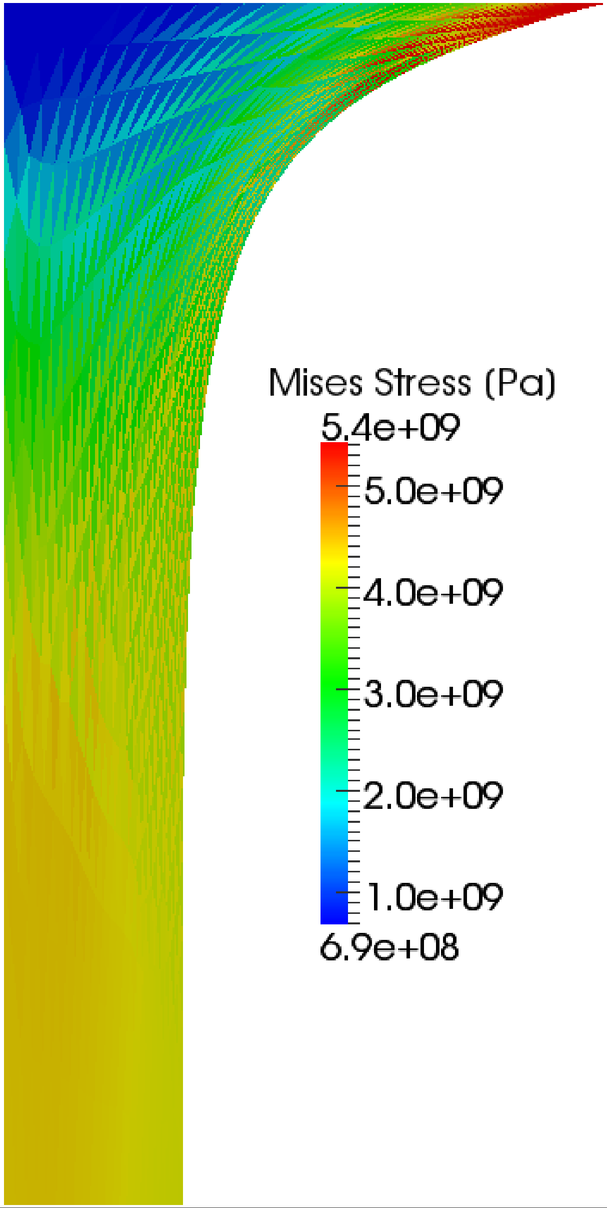
without Rezoning

Tension of 2D Brick

Proposed Method (59 Times Rezoning)



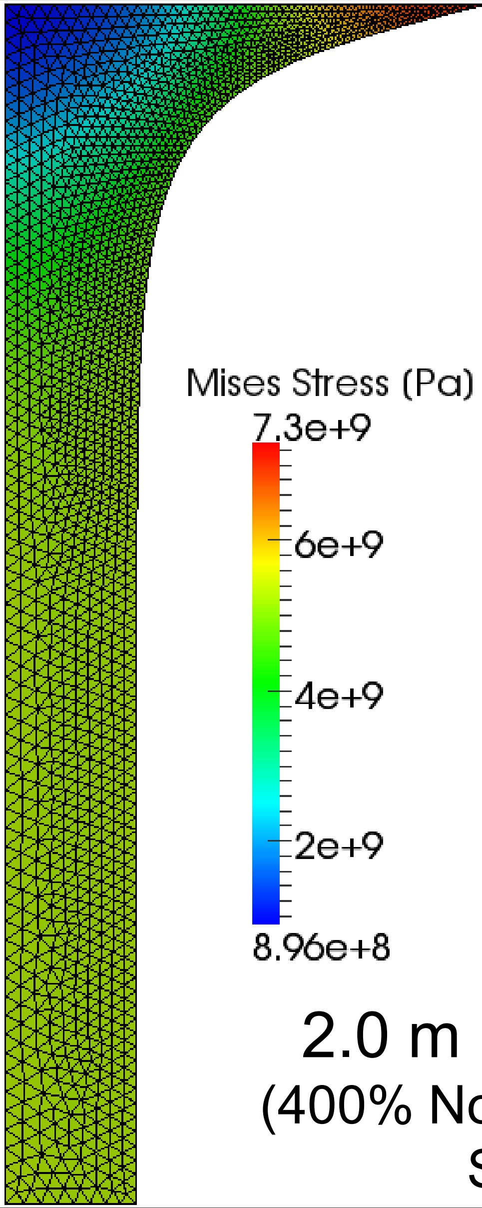
1.5 m Disp.
(300% Nominal Strain)



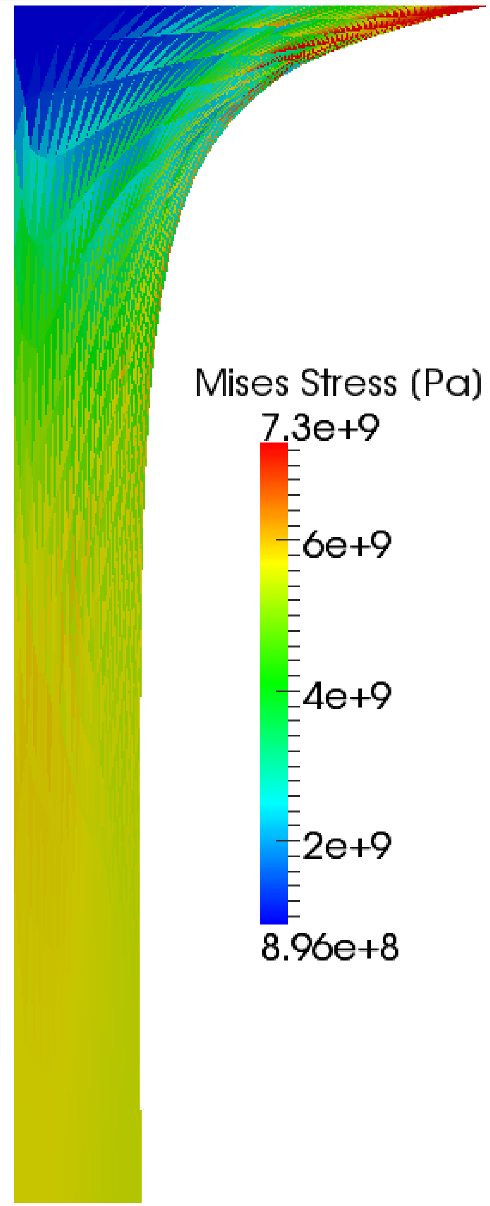
without Rezoning

Tension of 2D Brick

Proposed Method (79 Times Rezoning)

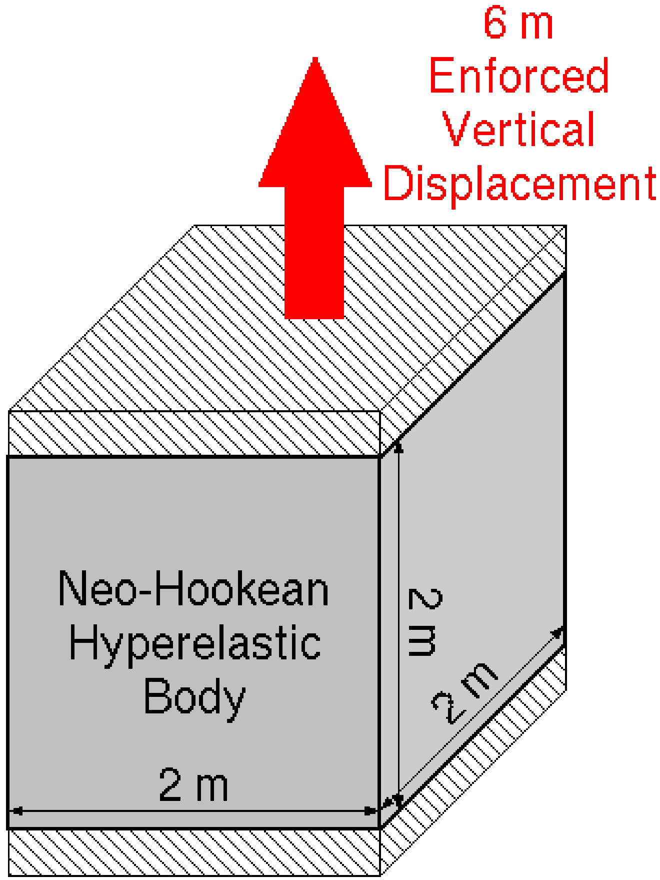


2.0 m Disp.
(400% Nominal Strain)



without Rezoning

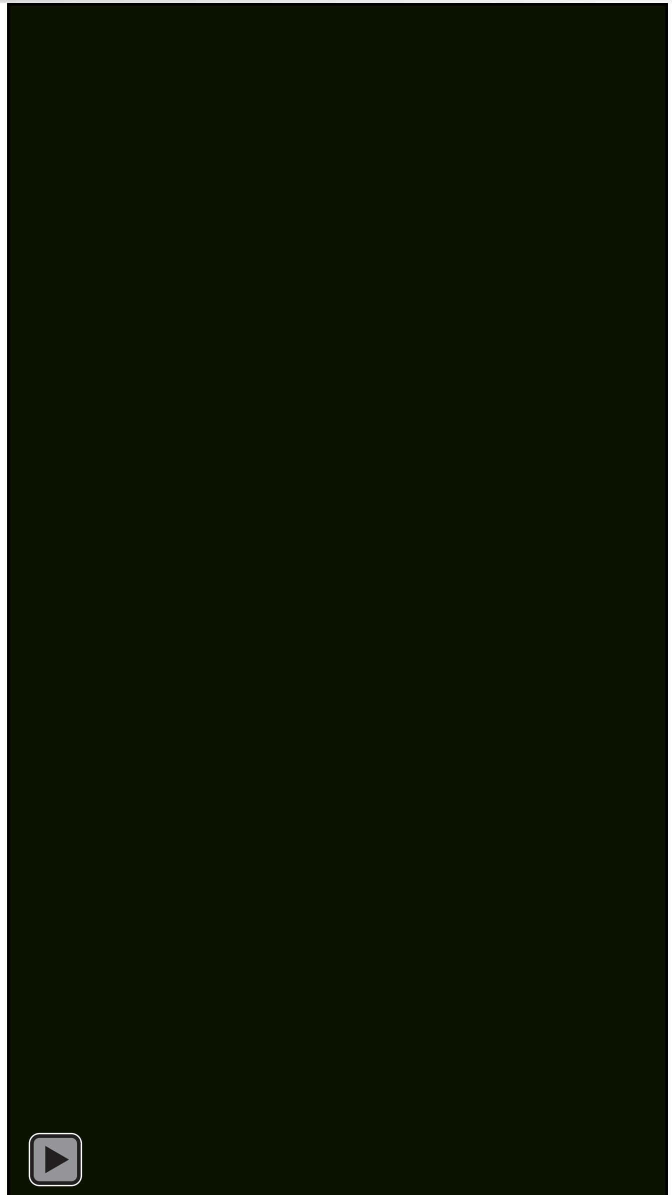
Tension of 3D Cube



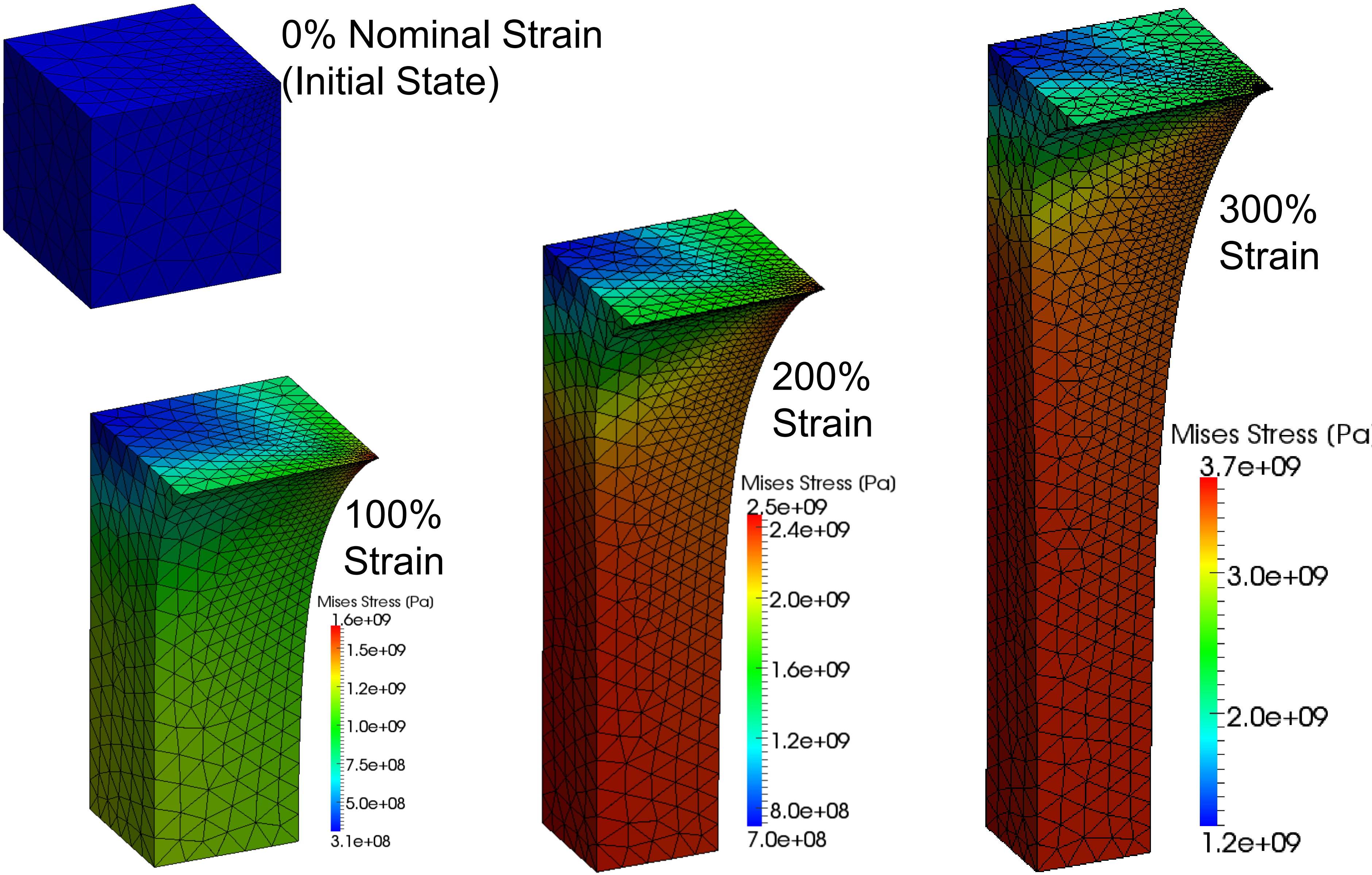
300%
Nominal Strain
at the Final State

- Static, 3D
- 1/8 model of the cube
- All 1st order tetrahedral elements
- Global rezoning every 10 timesteps

Tension of 3D Cube



Tension of 3D Cube



Summary and Future Work

Summary

- A new **implicit** FE rezoning method for severely large deformation analysis is proposed.
- It solves the IEE instead of the standard EE.
- It maps f^{ext} in addition to the other states.
- **Its accuracy and stability** are demonstrated.

Future Work

- More V&V
- SFEM implementation
- Apply to contact forming, crack propagation, etc.

Appendix

Mapping of f^{ext}

<u>Boil down to the following minimization problem:</u>

■ Unknown

nodal f^{ext} on the new mesh surface

■ Cost Function

$\sum$ || {surface traction on the new mesh face}

– {surface traction on the old mesh face} ||^2

■ Constraints

- $\sum$ {new nodal f^{ext}} = $\sum$ {old nodal f^{ext}}
- $\sum$ {new nodal $x \times f^{\text{ext}}$} = $\sum$ {old nodal $x \times f^{\text{ext}}$}

Solve it with Lagrange multiplier method

Derivation of Stiffness Matrix (1/2)

- Relation between $\dot{\Pi}_t$ and $\dot{T}$:

$$\dot{\boldsymbol{\Pi}}_t \equiv \dot{\boldsymbol{T}} + \mathrm{tr}(\boldsymbol{L})\boldsymbol{T} - \boldsymbol{L}\boldsymbol{T}$$

- Relation between $\dot{\square}$ and Jaumann rate:

$$\dot{\boldsymbol{T}} \equiv \mathring{\boldsymbol{T}} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W}$$

- Erasing $\dot{T}$:

$$\dot{\boldsymbol{\Pi}}_t^T = \mathring{\boldsymbol{T}} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W} + \mathrm{tr}(\boldsymbol{L})\boldsymbol{T} - \boldsymbol{T}\boldsymbol{L}^T$$

- Constitutive equation (e.g. Hencky's):

$$\mathring{\boldsymbol{T}} = \boldsymbol{C}_{\mathrm{L}} : \boldsymbol{D}$$

- Erasing $\mathring{\boldsymbol{T}}$:

$$\dot{\boldsymbol{\Pi}}_t^T = \boldsymbol{C}_{\mathrm{L}} : \boldsymbol{D} + \boldsymbol{W}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{W} + \mathrm{tr}(\boldsymbol{L})\boldsymbol{T} - \boldsymbol{T}\boldsymbol{L}^T$$

Derivation of Stiffness Matrix (2/2)

- Rewrite in matrix form:

$$\{\dot{\Pi}_t^T\} = [C_\mathrm{L}]\{D\} + [C_\mathrm{N}]\{L\}$$

where

$$[C_\mathrm{N}] = \begin{bmatrix} 0 & T_{xx} & T_{xx} & 0 & 0 & -T_{xy} & 0 & -T_{zx} & 0 \\ T_{yy} & 0 & T_{yy} & -T_{xy} & 0 & 0 & 0 & 0 & -T_{yz} \\ T_{zz} & T_{zz} & 0 & 0 & -T_{zx} & 0 & -T_{yz} & 0 & 0 \\ T_{xy} & 0 & T_{xy} & \frac{T_{yy}-T_{xx}}{2} & \frac{T_{yz}}{2} & \frac{-T_{yy}-T_{xx}}{2} & \frac{-T_{zx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{zx}}{2} \\ T_{zx} & T_{zx} & 0 & \frac{T_{yz}}{2} & \frac{T_{zz}-T_{xx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{xy}}{2} & \frac{-T_{zz}-T_{xx}}{2} & \frac{-T_{xy}}{2} \\ 0 & T_{xy} & T_{xy} & \frac{-T_{yy}-T_{xx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{yy}+T_{xx}}{2} & \frac{T_{zx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{zx}}{2} \\ T_{yz} & T_{yz} & 0 & \frac{-T_{zx}}{2} & \frac{-T_{xy}}{2} & \frac{T_{zx}}{2} & \frac{T_{zz}-T_{yy}}{2} & \frac{-T_{xy}}{2} & \frac{-T_{zz}-T_{yy}}{2} \\ 0 & T_{zx} & T_{zx} & \frac{-T_{yz}}{2} & \frac{-T_{zz}-T_{xx}}{2} & \frac{-T_{yz}}{2} & \frac{-T_{xy}}{2} & \frac{-T_{zz}+T_{xx}}{2} & \frac{T_{xy}}{2} \\ T_{yz} & 0 & T_{yz} & \frac{-T_{zx}}{2} & \frac{-T_{xy}}{2} & \frac{-T_{zx}}{2} & \frac{-T_{zz}-T_{yy}}{2} & \frac{T_{xy}}{2} & \frac{-T_{zz}+T_{yy}}{2} \end{bmatrix}$$

- Stiffness Matrix

$$[K^+] = \sum_{e \in \mathbb{E}} \int_{\Omega_e^+} [B_\mathrm{L}^+]^T [C_\mathrm{L}][B_\mathrm{L}^+] + [B_\mathrm{N}^+]^T [C_\mathrm{N}][B_\mathrm{N}^+] \, \mathrm{d}\Omega$$