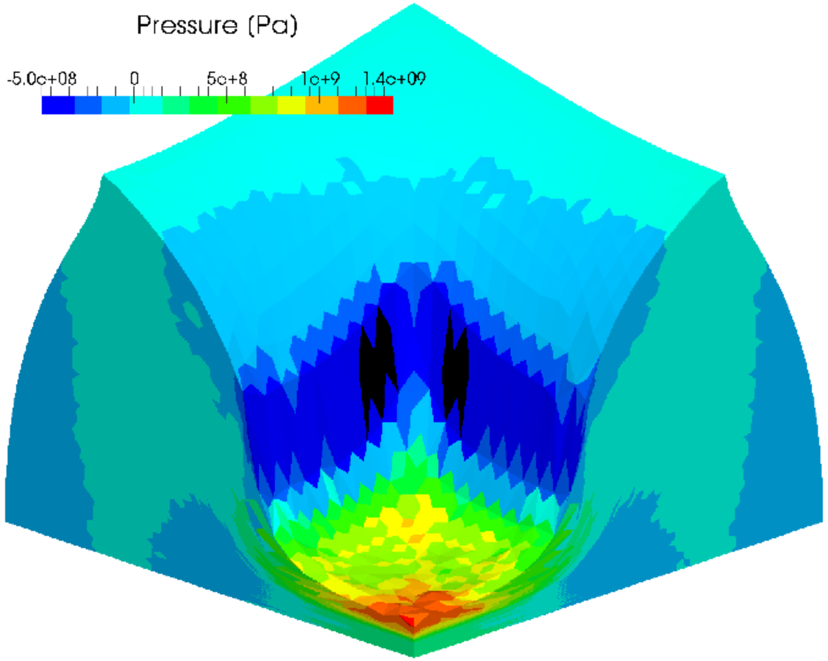
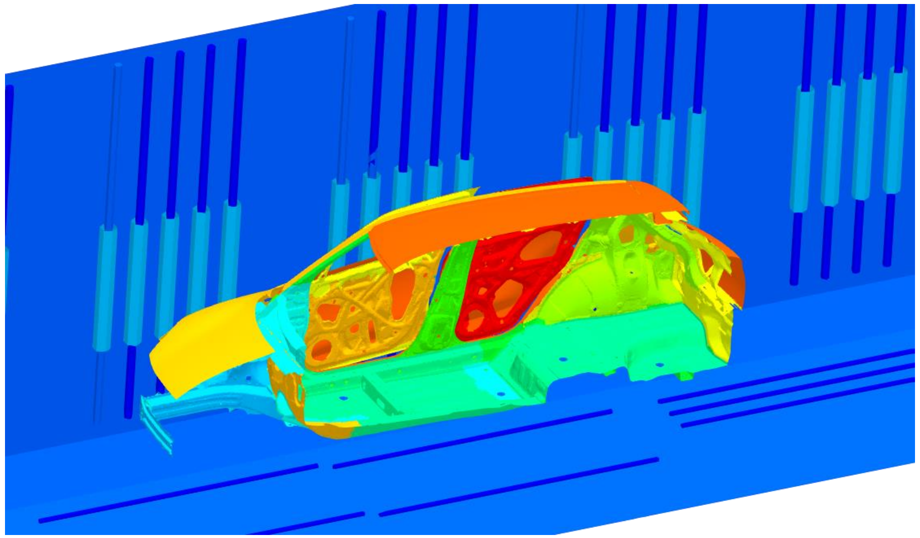
A Brief Introduction to Smoothed Finite Element Method (S-FEM)





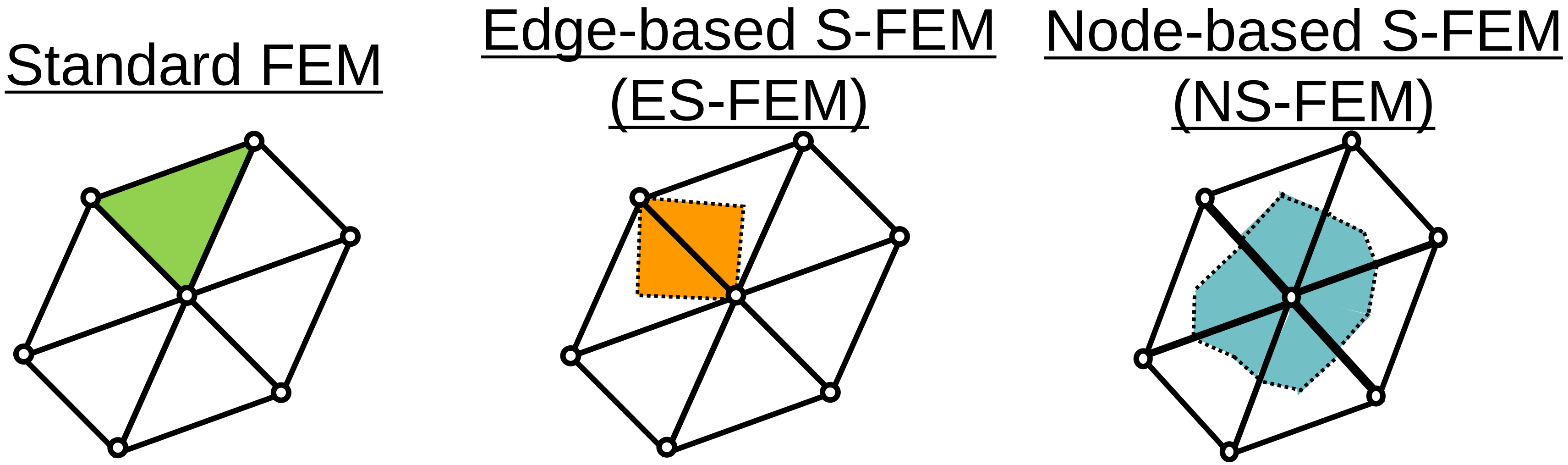
Yuki ONISHI (Tokyo Institute of Technology)

Quick Review of S-FEM

What is S-FEM?

- ***Smoothed*** finite element method (**S-FEM**) is a relatively new FE formulation proposed in 2006.
- S-FEM is one of the **gradient (strain) smoothing** techniques.
- There are **many kinds of S-FEMs** depending on the scheme of smoothing.
- There are a few *classical* S-FEMs depending on the smoothing domain.

For example, in a 2D triangular mesh:



Each colored area shows the domain for gradient smoothing.

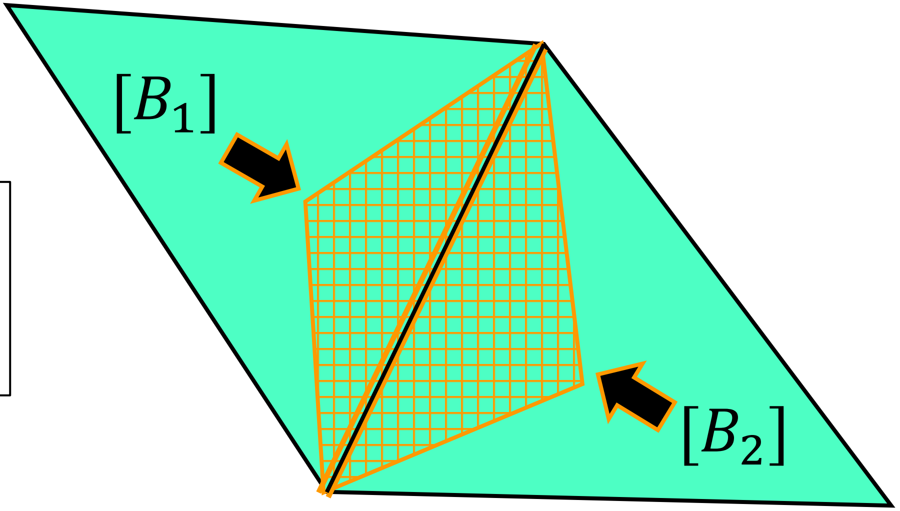
e.g.) Brief of **ES-FEM**

Let me explain in 2D for simplicity

Let us consider a mesh with only two 3-node triangular cells.

- Calculate $[B](=\mathrm{d}\boldsymbol{N}/\mathrm{d}\boldsymbol{x})$ at each cell as usual.
- Distribute each $[B]$ to the connecting edge with an area weight and build $[{}^{\mathrm{Edge}}B]$.
- Calculate strain ($\boldsymbol{\varepsilon}$), Cauchy stress ($\boldsymbol{\sigma}$) and nodal internal force $\{f^{\mathrm{int}}\}$ in each edge smoothing domain with $[{}^{\mathrm{Edge}}B]$.

As if putting
a Gauss point
on each edge center



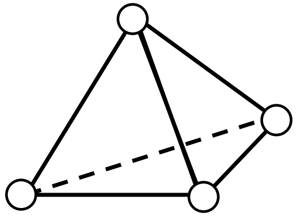
$[{}^{\mathrm{Edge}}B]$

↳ ${}^{\mathrm{Edge}}\boldsymbol{\varepsilon}$, ${}^{\mathrm{Edge}}\boldsymbol{\sigma}$, $\{{}^{\mathrm{Edge}}f^{\mathrm{int}}\}$ etc.

What are the major benefits of S-FEM?

1. **Super-linear mesh convergence rate with T4 mesh.**
   (Almost same rate as 2nd-order elements with T4 mesh.)
2. **Shear locking free with ES-FEM-T4.**
   (Good accuracy with T4 mesh in solid mechanics.)
3. **Volumetric locking free with NS-FEM-T4.**
   (Key technique for rubber-like nearly incompressible solid.)
4. **Little accuracy loss with skewed meshes.**
   (No problem with complex geometry or severe deformation.)
5. **No increase in DOF.**
   (Purely displacement-based formulation.)
6. **Easy to code.**
   (keeping away from mixed variational formulations.)

T4: 4-node Tetrahedra



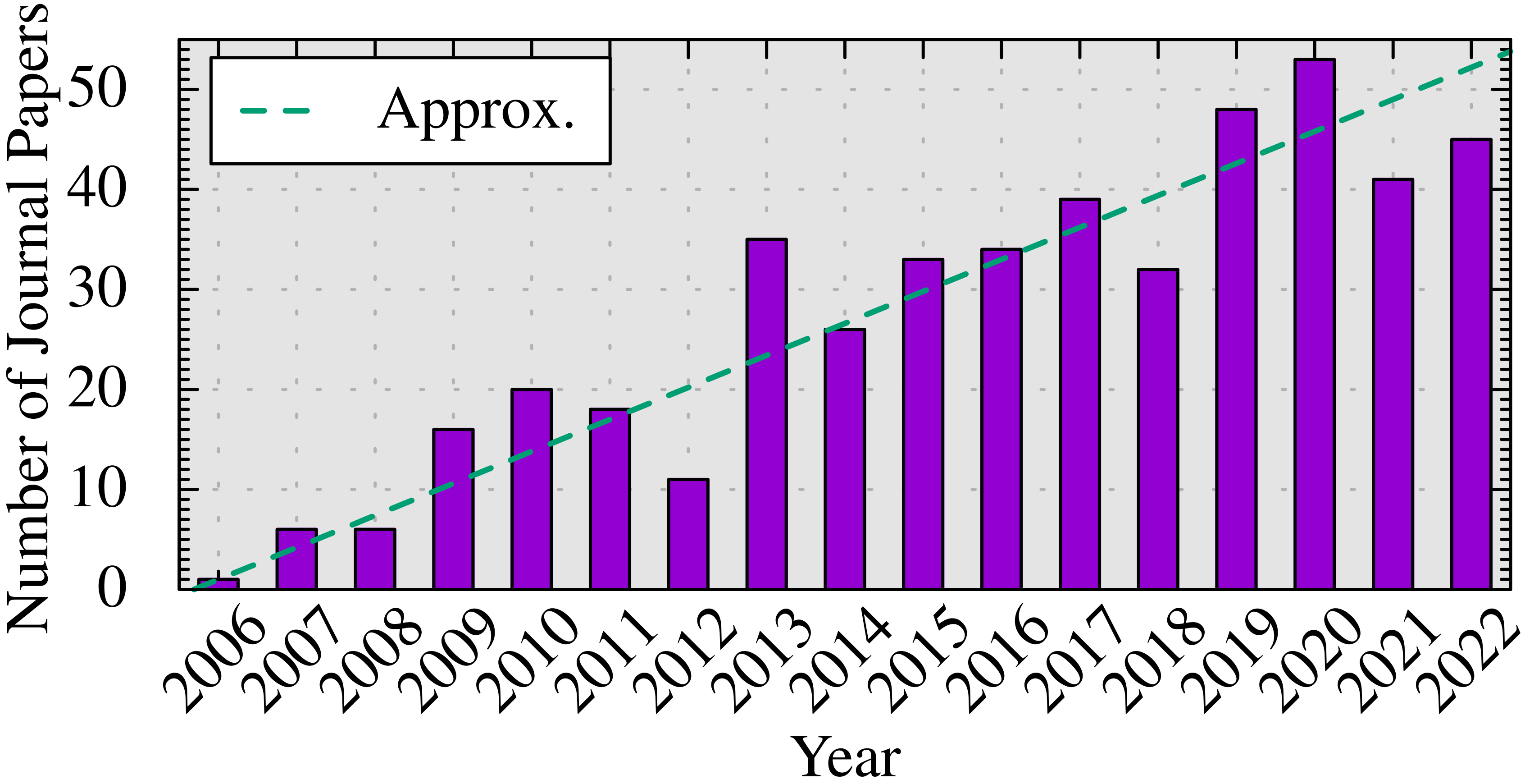
T4

S-FEM is a powerful method suitable for practical industrial applications.

How popular is S-FEM?

Number of journal papers whose **title** contains "**smoothed finite element**":

inquired at
Google Scholar



Two active researchers in Germany:

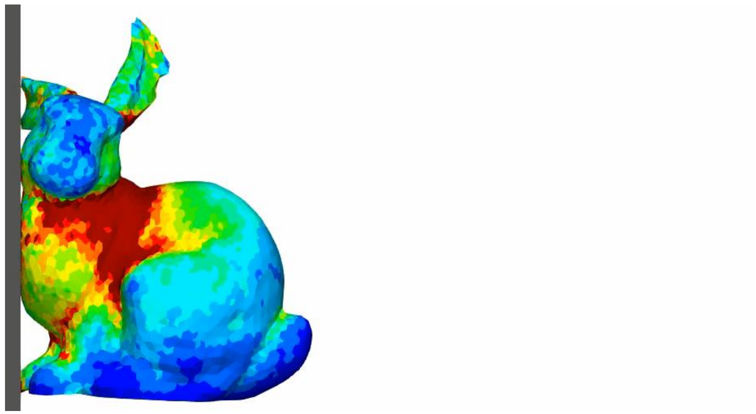
- Prof. Günther Meschke
  in Ruhr University Bochum
- Denisa Martonová
  in Friedrich-Alexander-Universität

The attraction of S-FEM is expanding continuously.

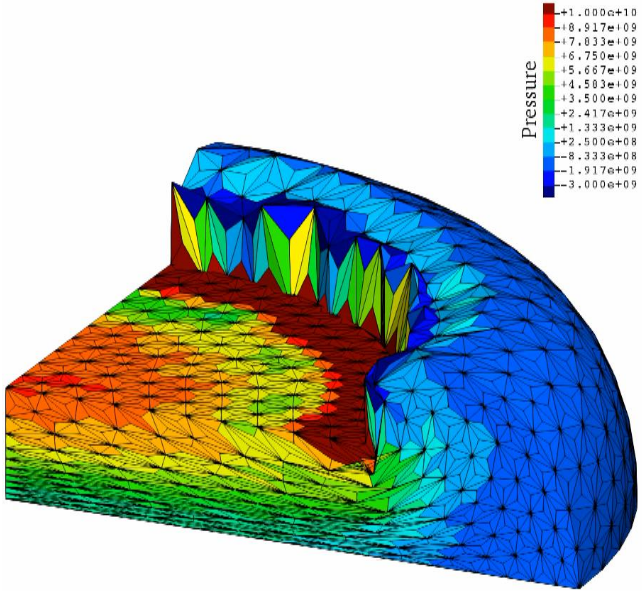
Applications of S-FEM-T4 in Our Lab

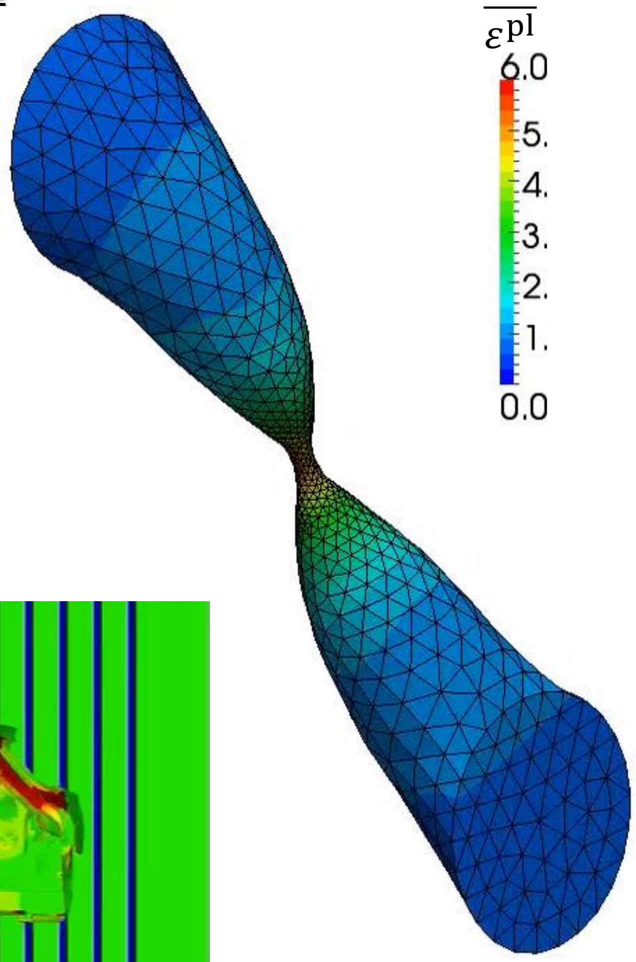
- **Large deformation solid mechanics (still in academic research)**

Dynamic Explicit

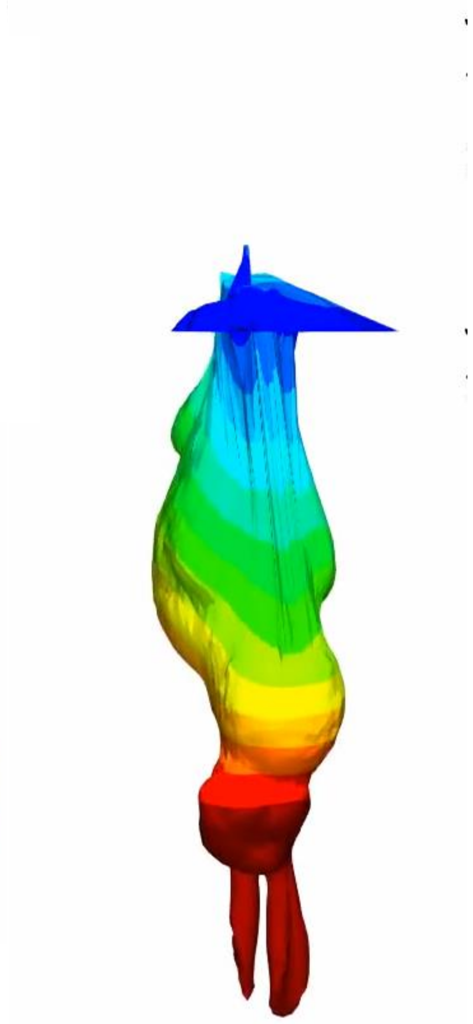


Static Implicit



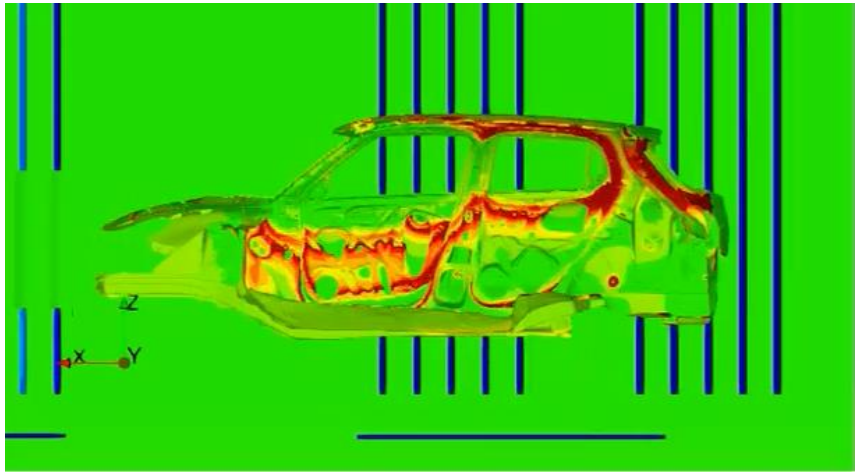


Viscous Implicit



- **Electrostatic (already in commercial use)**

Carbody Electro Deposition



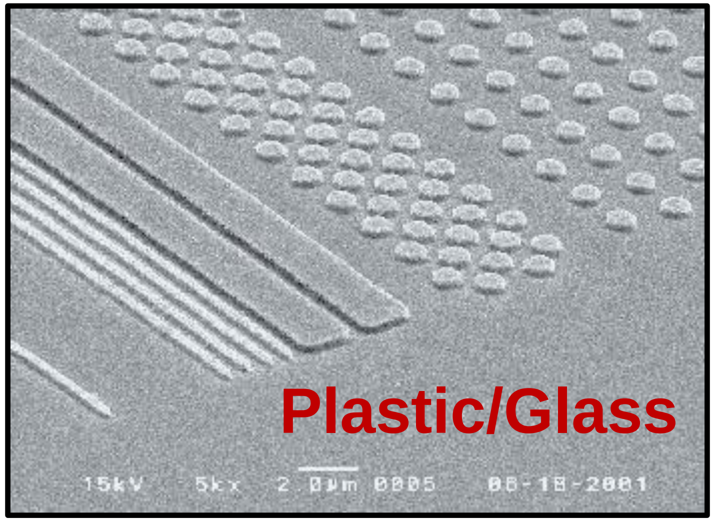
Today, I'll focus on **static implicit** analyses and introduce the latest studies on S-FEM-T4 briefly.

Motivation & Objective of Our Latest Study

Motivation

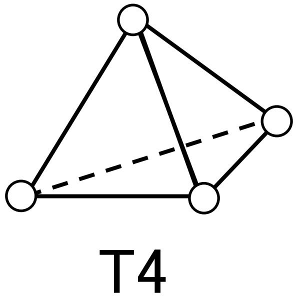
What we want to do:

- Solve **severe large deformation** analyses accurately and robustly.
- Treat complex geometries with **tetrahedral meshes**.
- Consider **nearly incompressible materials $(\boldsymbol{\nu \simeq 0.5})$**.
- Support **contact** problems.
- Handle **auto re-meshing**.

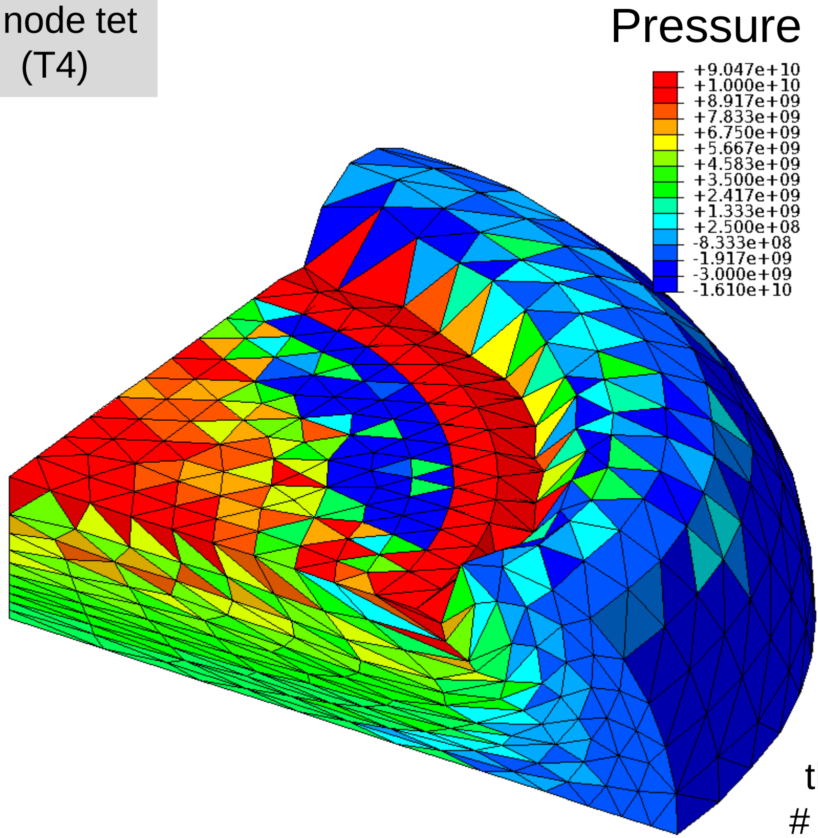


Issues in Conventional FE (ABAQUS)

e.g.) Barreling of Rubber Cylinder Neo-Hookean ***hyperelastic*** body with $\nu_{\mathrm{ini}} = \mathbf{0.49}$

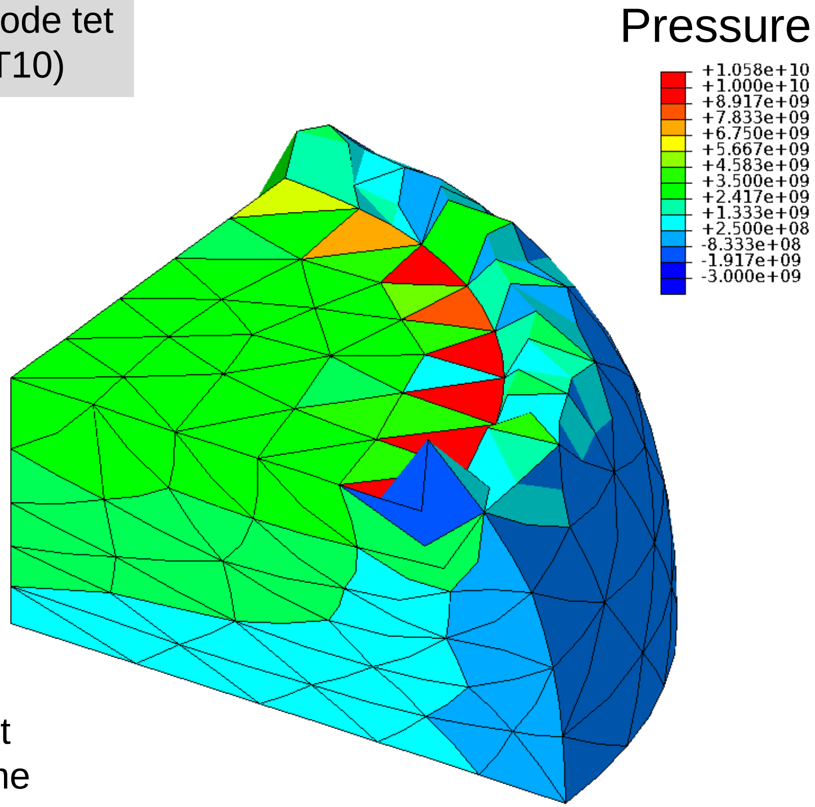


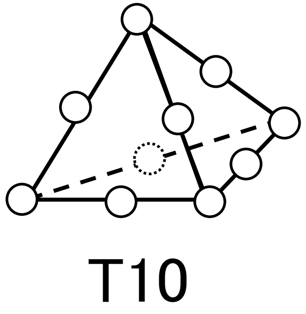
4 node tet
(T4)



Almost
the same
# of nodes.

10 node tet
(T10)





ABQUS C3D4H

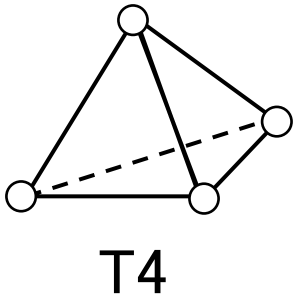
- ✓ No volumetric locking.
- ✗ Pressure checkerboarding.
- ✗ Shear locking & Corner locking.

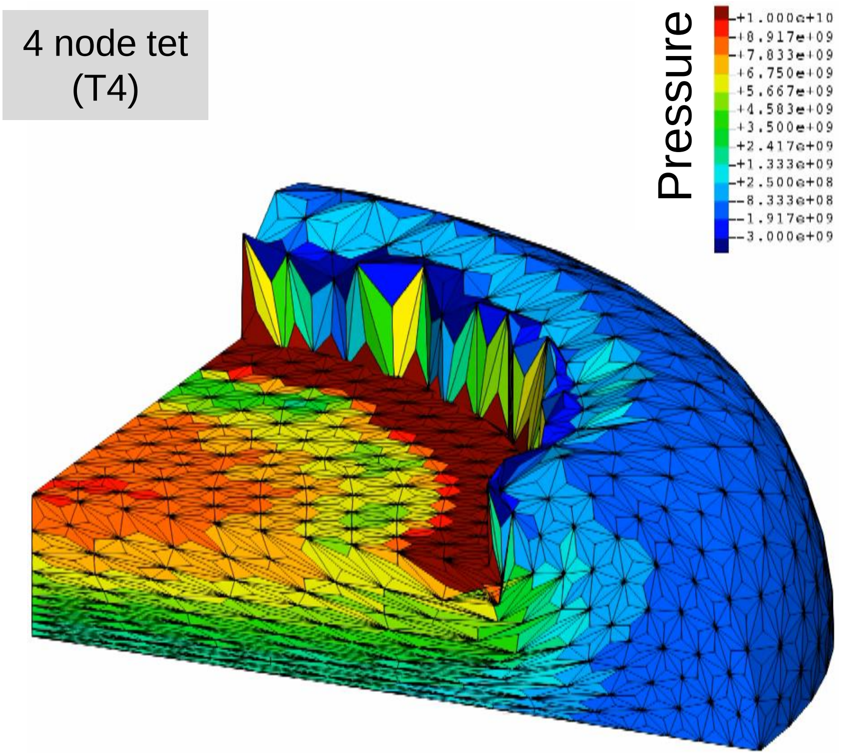
ABAQUS C3D10MH

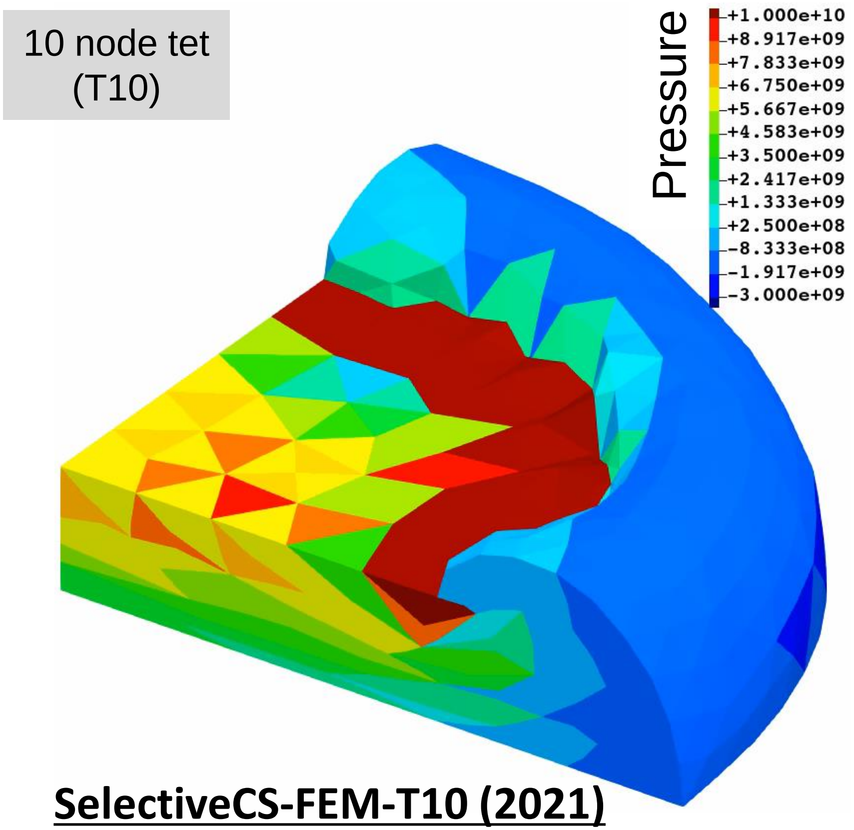
- ✓ No shear/volumetric locking.
- ✗ Short lasting (weak to severe deformation).
- ✗ Low interpolation accuracy.

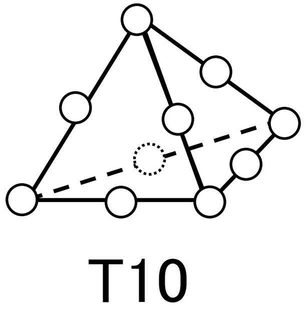
Our Approach using S-FEM

e.g.) Barreling of Rubber Cylinder Neo-Hookean ***hyperelastic*** body with $\nu_{\text{ini}} = 0.49$









More than 10 times slower than FEM-T4, but no good idea for speed-up…

F-barES-FEM-T4 (2017)

- ✓ No shear/volumetric locking.
- ✓ Less pressure checkerboarding.
- ✓ Less corner locking. Long lasting.
- ✓ No oscillation in deviatoric stress.
- ✗ Long CPU time. Incompatible w/ FE.

SelectiveCS-FEM-T10 (2021)

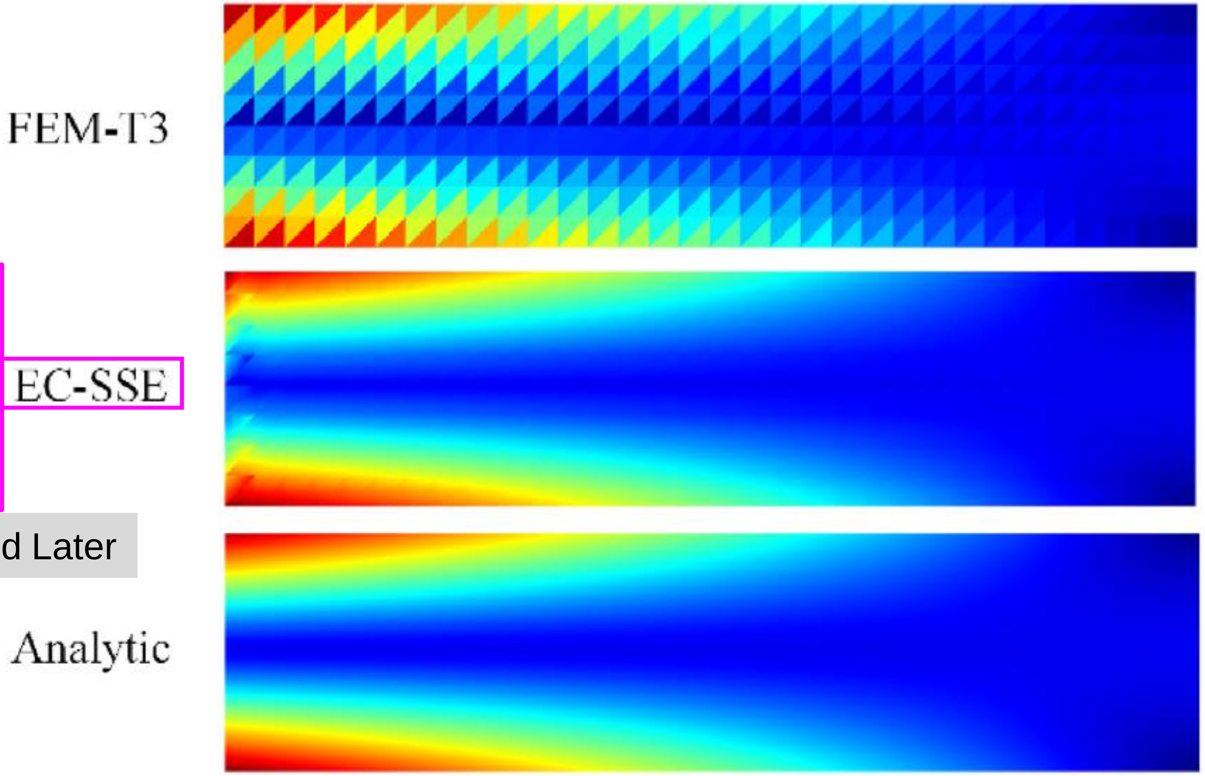
- ✓ No shear/volumetric locking.
- ✓ Less pressure checkerboarding.
- ✓ Less corner locking. Long lasting.
- ✗ Major oscillation in deviatoric stress.
- ✓ Same CPU time. Compatible w/ FE.

Cannot suppress stress oscillation, but no good idea for accuracy improvement…

Birth of a New-generation S-FEM, EC-SSE, in 2022

Comparison of Mises stress dist. in cantilever bending analyses in 2D

T. Jinsong *et al*., Euro. J. Mech. /A, v95, 2022.



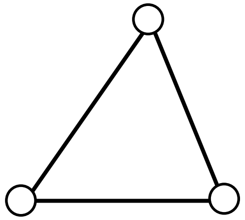
Edge
Center-based
Strain
Smoothing
Element

Detailed Later

- ➢ Step-like stress dist. (poor).
- ➢ Shear locking.

- ➢ Linear stress dist. (very good) using the same T3 mesh. So close to analytical solution.
- ➢ No shear locking.

This breakthrough should be called "**S-FEM 2.0**".



T3

Poisson's Ratio: $\nu = 0.3$

EC-SSE is an excellent formulation for compressible solids;
but when $\nu \simeq 0.5$, EC-SSE has **volumetric locking** and **pressure checkerboarding**.
Therefore, EC-SSE is NOT directly applicable to nearly incompressible solids.

Objective

Objective

Develop a new S-FEM formulation to extend EC-SSE to nearly incompressible large deformation analysis

Strategy

Use the selective reduced integration (SRI)

- Use EC-SSE for the deviatoric part,
- Use NS-FEM for the volumetric part, and
- Combine them with SRI.

EC-SSE-SRI

Method

Introduction to ES-FEM, NS-FEM, EC-SSE, and EC-SSE-SRI

Brief of ES-FEM

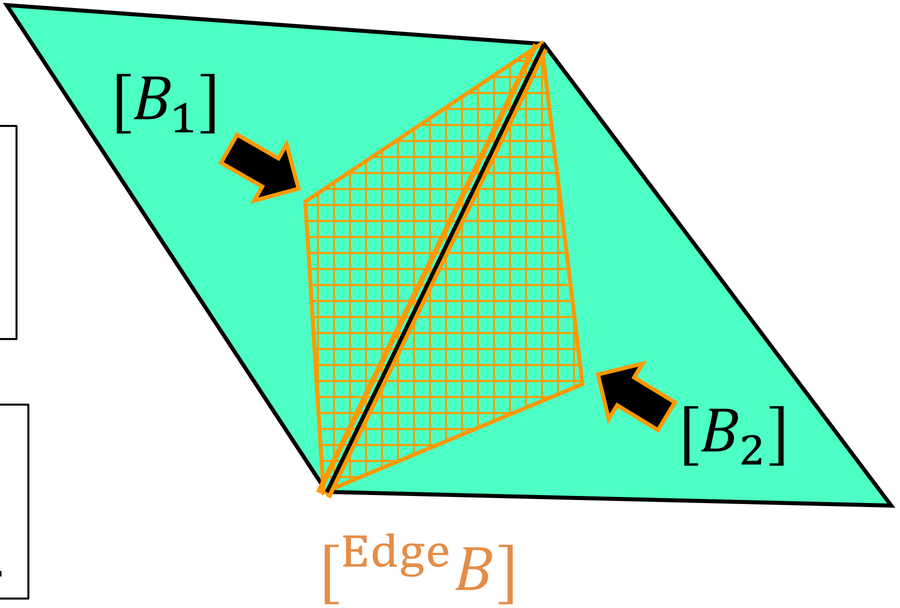
Let me explain in 2D for simplicity

Let us consider a mesh with only two 3-node triangular cells.

- Calculate $[B](= \mathrm{d}\boldsymbol{N}/\mathrm{d}\boldsymbol{x})$ at each cell as usual.
- Distribute each $[B]$ to the connecting edge with an area weight and build $[{}^{\mathrm{Edge}}B]$.
- Calculate deformation gradient ($\boldsymbol{F}$), Cauchy stress ($\boldsymbol{\sigma}$) and nodal internal force $\{f^{\mathrm{int}}\}$ in each edge smoothing domain with $[{}^{\mathrm{Edge}}B]$.

As if putting a Gauss point on each edge center

Strain distribution is piecewise constant in each smoothing domain.



↳ ${}^{\mathrm{Edge}}\boldsymbol{F}$, ${}^{\mathrm{Edge}}\boldsymbol{\sigma}$, $\{{}^{\mathrm{Edge}}f^{\mathrm{int}}\}$ etc.

No shear locking.

Cannot avoid volumetric locking & pressure checkerboarding.

Brief of **NS-FEM**

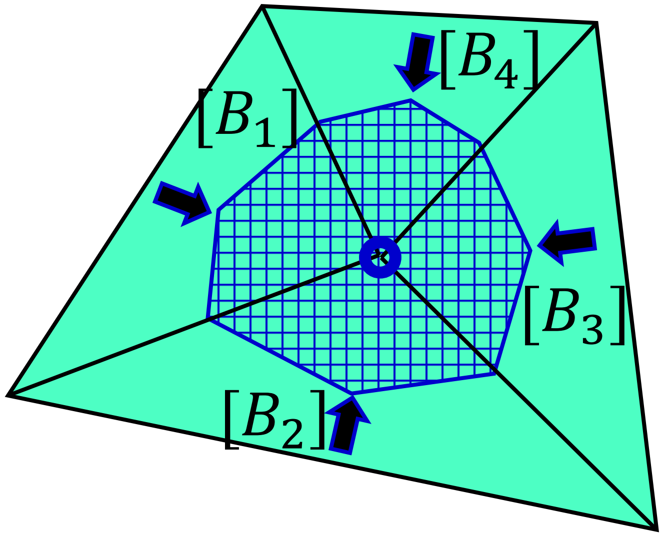
Let us consider a mesh with only four 3-node triangular cells.

Let me
explain in 2D
for simplicity

- Calculate $[B](=\mathrm{d}\boldsymbol{N}/\mathrm{d}\boldsymbol{x})$ at each cell as usual.
- Distribute each $[B]$ to the connecting node with an area weight and build $[{}^{\mathrm{Node}}B]$.
- Calculate deformation gradient $(\boldsymbol{F})$, Cauchy stress $(\boldsymbol{\sigma})$ and nodal internal force $\{f^{\mathrm{int}}\}$ in each nodal smoothing domain with $[{}^{\mathrm{Node}}B]$.

As if putting
a Gauss point
on each node center

Strain distribution is
piecewise constant
in each smoothing domain.



No shar/volumetric locking.
Less pressure checkerboarding

Cannot avoid
spurious low-energy modes.

$[{}^{\mathrm{Node}}B]$

$\hookrightarrow$ ${}^{\mathrm{Node}}\boldsymbol{F}$, ${}^{\mathrm{Node}}\boldsymbol{\sigma}$, $\{{}^{\mathrm{Node}}f^{\mathrm{int}}\}$ etc.

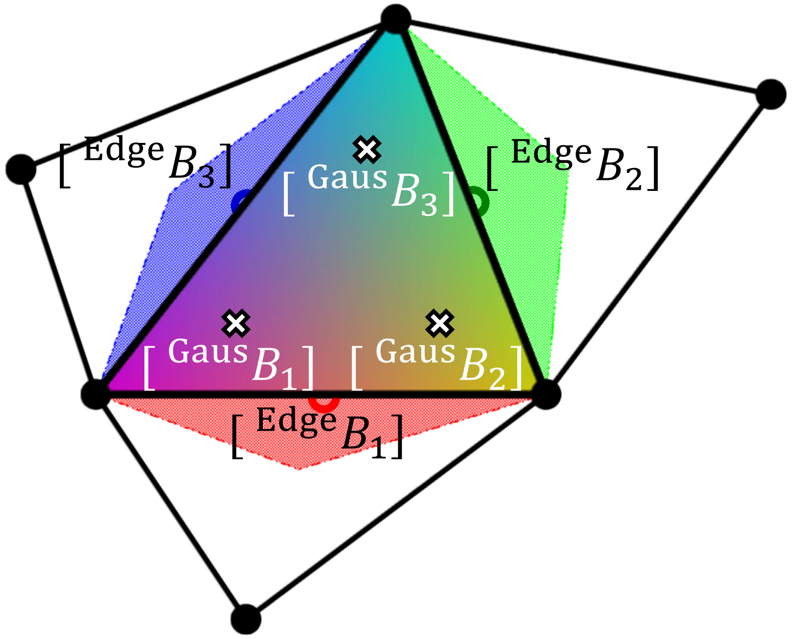
Brief of EC-SSE

- Make $[{}^{\mathrm{Edge}}B]$ s in the same procedure as ES-FEM.
- Consider each $[{}^{\mathrm{Edge}}B]$ is the value at the center of each edge, and **<u>assume $[B]$ is linearly distributed in each cell</u>**.
- Make three $[{}^{\mathrm{Gaus}}B]$ s in each cell as the extrapolation of the three $[{}^{\mathrm{Edge}}B]$ s.
- Calculate ${}^{\mathrm{Gaus}}\varepsilon$, ${}^{\mathrm{Gaus}}\sigma$ and $\{f^{\mathrm{int}}\}$ using each $[{}^{\mathrm{Gaus}}B]$ in the same manner as the 2nd -order element.

Let me explain in 2D for simplicity

Conducting strain smoothing twice, the strain/stress are evaluated at each Gauss point.

Strain distribution is <u>piecewise-linear in each cell</u> and is **<u>continuing at every edge center</u>**.



- No shear locking with T3/T4 mesh.
- Fast mesh convergence rate in strain/stress as an 2nd –order element.

- Cannot avoid volumetric locking and pressure checkerboarding

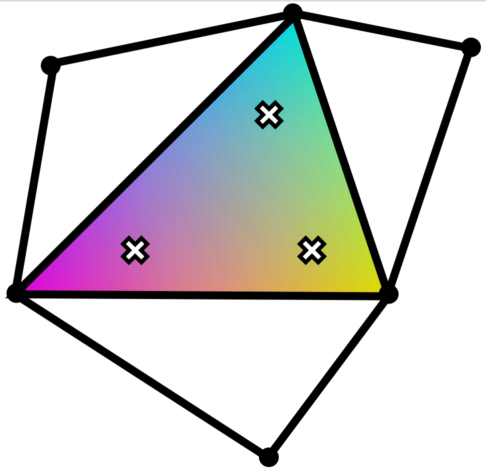
Brief of EC-SSE-SRI (Our Latest Method)

Apply the **selective reduced integration (SRI)** to EC-SSE to handle rubber-like solids

Let me explain in 2D for simplicity

(1) Calculate $\varepsilon^{\mathrm{dev}}$ at each Gauss point with EC-SSE

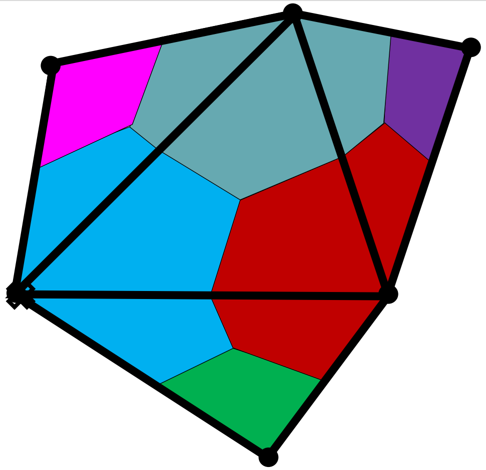
Deviatoric Part



(2) Calculate σ^{dev} at each Gauss point and its contribution to $\{f^{\mathrm{int}}\}$

(3) Calculate $\varepsilon^{\mathrm{vol}}$ at each node with NS-FEM

Volumetric Part



(4) Calculate σ^{hyd} at each node and its contribution to $\{f^{\mathrm{int}}\}$

Selective Reduced Integration (SRI)

(5) Assemble $\{f^{\mathrm{int}}\}$

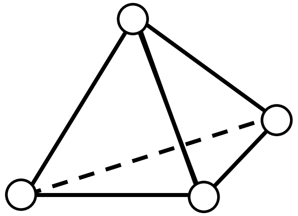
Deviatoric strain distribution is piecewise-linear in each cell and is **continuing at each edge center.**

No shear/volumetric locking.
Less pressure checkerboarding.

Brief of EC-SSE-SRI-T4 (in 3D)

Let me explain with text only

[Deviatoric Part]



- Make $\left[{}^{\mathrm{Edge}}B\right]$ s in the same procedure as ES-FEM.
- **Make $[{}^{\mathbf{Face}}\boldsymbol{B}]$ s by re-smoothing three $\left[{}^{\mathbf{Edge}}\boldsymbol{B}\right]$s per face.**
- Consider each $[{}^{\mathbf{Face}}\boldsymbol{B}]$ is the value at the center of each **face**, and assume $[B]$ is linearly distributed in each cell.
- Make **four** $[{}^{\mathbf{Gaus}}\boldsymbol{B}]$ s in each cell as the extrapolation of the **four** $[{}^{\mathbf{Face}}\boldsymbol{B}]$ **s**.
- Calculate ${}^{\mathrm{Gaus}}\varepsilon_{\mathrm{dev}}$, ${}^{\mathrm{Gaus}}\sigma_{\mathrm{dev}}$ and $\{f_{\mathrm{dev}}^{\mathrm{int}}\}$ using each $[{}^{\mathrm{Gaus}}B]$, like the 2nd -order element.

[Volumetric Part]

- Make $\left[{}^{\mathrm{Node}}B\right]$s in the same procedure as NS-FEM.
- Calculate ${}^{\mathrm{Node}}\varepsilon_{\mathrm{vol}}$, ${}^{\mathrm{Node}}\sigma_{\mathrm{hyd}}$ and $\{f_{\mathrm{vol}}^{\mathrm{int}}\}$ using each $[{}^{\mathrm{Node}}B]$.

[SRI]

- Make $\{f^{\mathrm{int}}\} = \{f_{\mathrm{dev}}^{\mathrm{int}}\} + \{f_{\mathrm{vol}}^{\mathrm{int}}\}$.

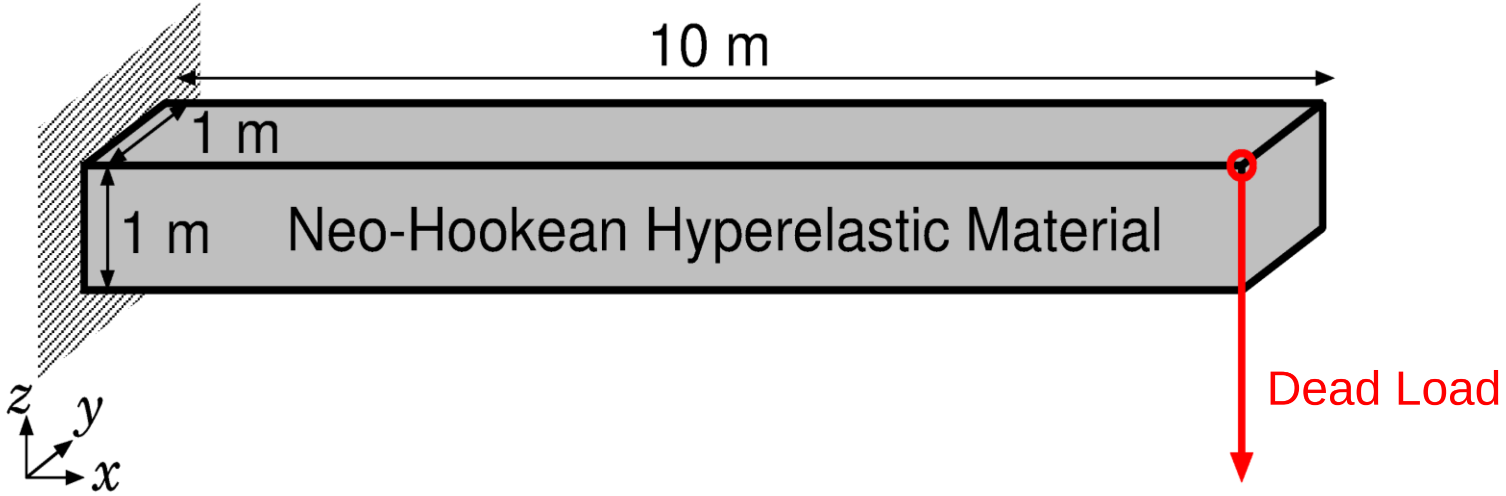
Result & Discussion

Demonstration of EC-SSE-SRI-T4 in 3D and Evaluation of CPU Cost

Static Implicit

Bending of Rubber Cantilever

Outline

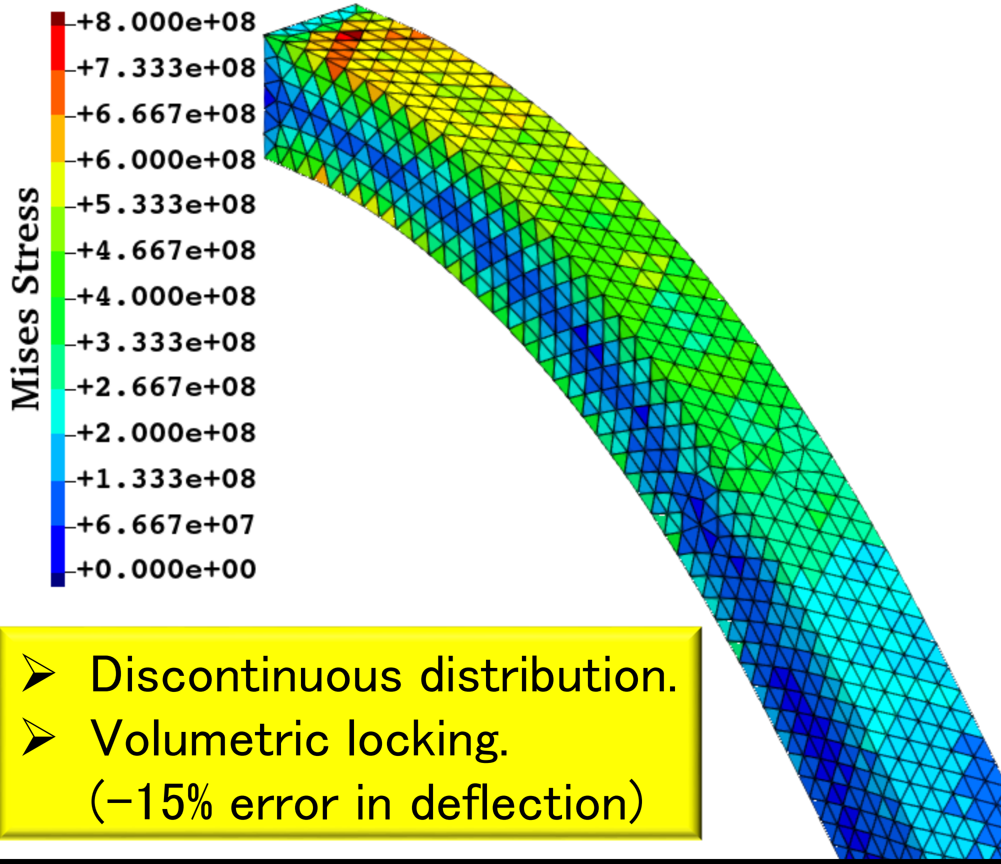


- 10 x 1 x 1 m cantilever.
- Dead load applied to the tip node.
- Neo-Hookean hyperelastic material, $E_{\mathrm{ini}} = 6\ \mathrm{GPa}$, $\boldsymbol{\nu}_{\mathbf{ini}} = \mathbf{0.49}$.
- A large deflection analysis with $u_z = -6.5\ \mathrm{m}$ at the final state.
- Compared the results of **ABAQUS C3D4** and EC-SSE-SRI-T4.

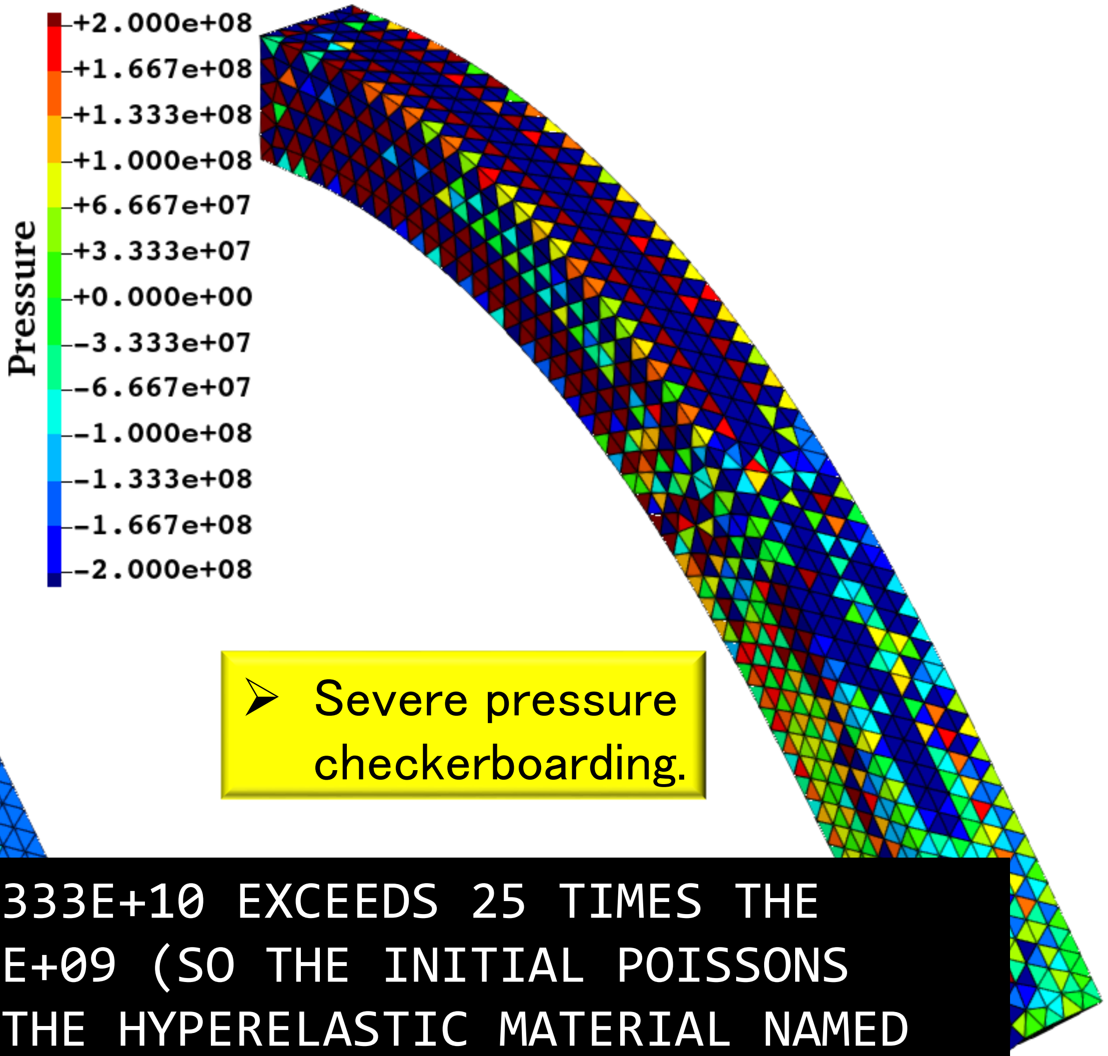
Static
Implicit

Bending of Rubber Cantilever

Results of ABAQUS C3D4 (Final State)



- Discontinuous distribution.
- Volumetric locking.
  (−15% error in deflection)



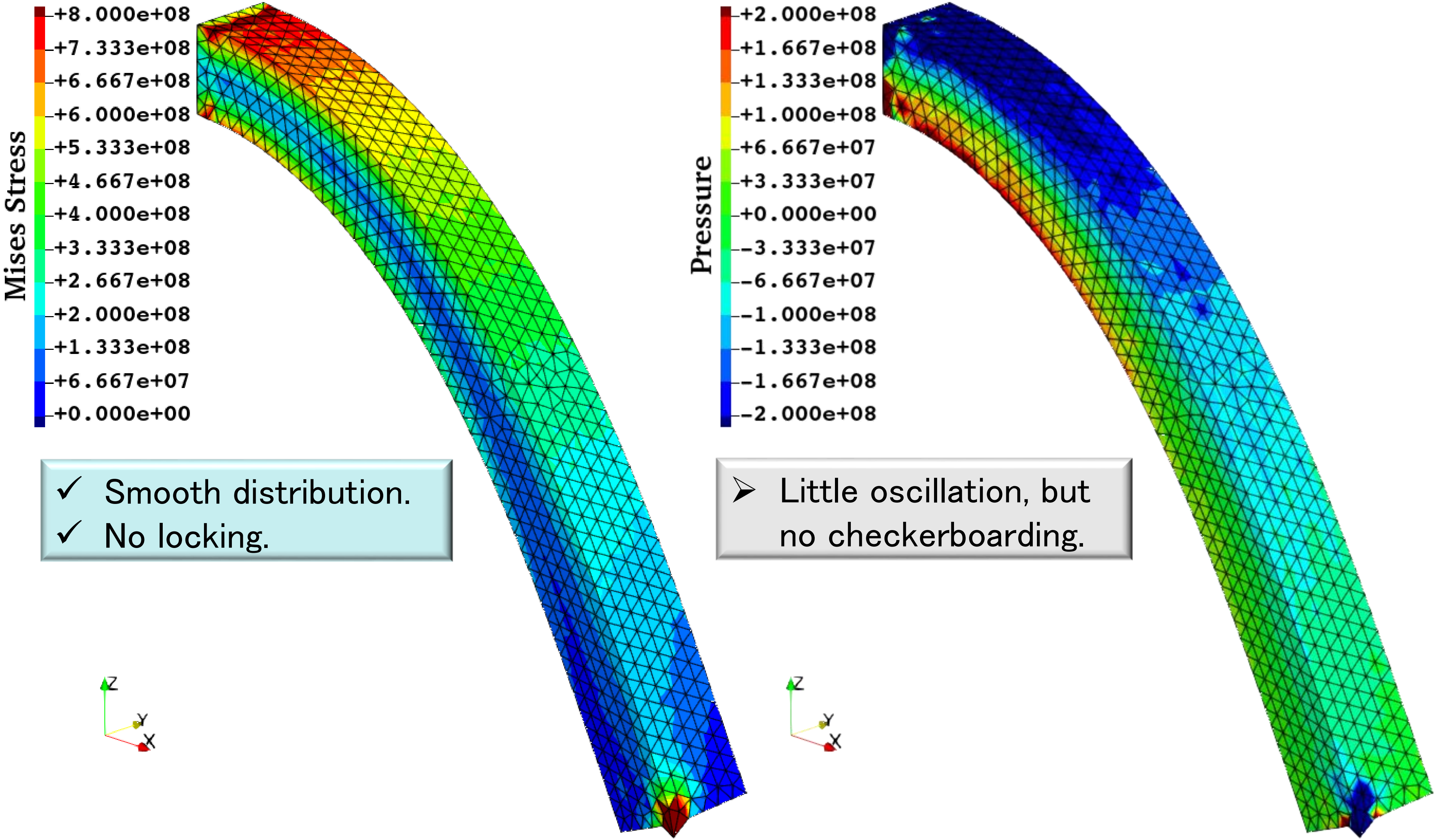
- Severe pressure checkerboarding.

ABAQUS dat file:

```
***WARNING: THE INITIAL BULK MODULUS OF 9.93333E+10 EXCEEDS 25 TIMES THE
            INITIAL SHEAR MODULUS OF 2.00000E+09 (SO THE INITIAL POISSONS
            RATIO 0.49000 EXCEEDS 0.48) FOR THE HYPERELASTIC MATERIAL NAMED
            MATERIAL-1. HOWEVER, A HYBRID TYPE ELEMENT IS NOT USED. THIS MAY
            CAUSE CONVERGENCE PROBLEMS.
```

Bending of Rubber Cantilever

Results of
EC-SSE
-SRI-T4
(Final State)



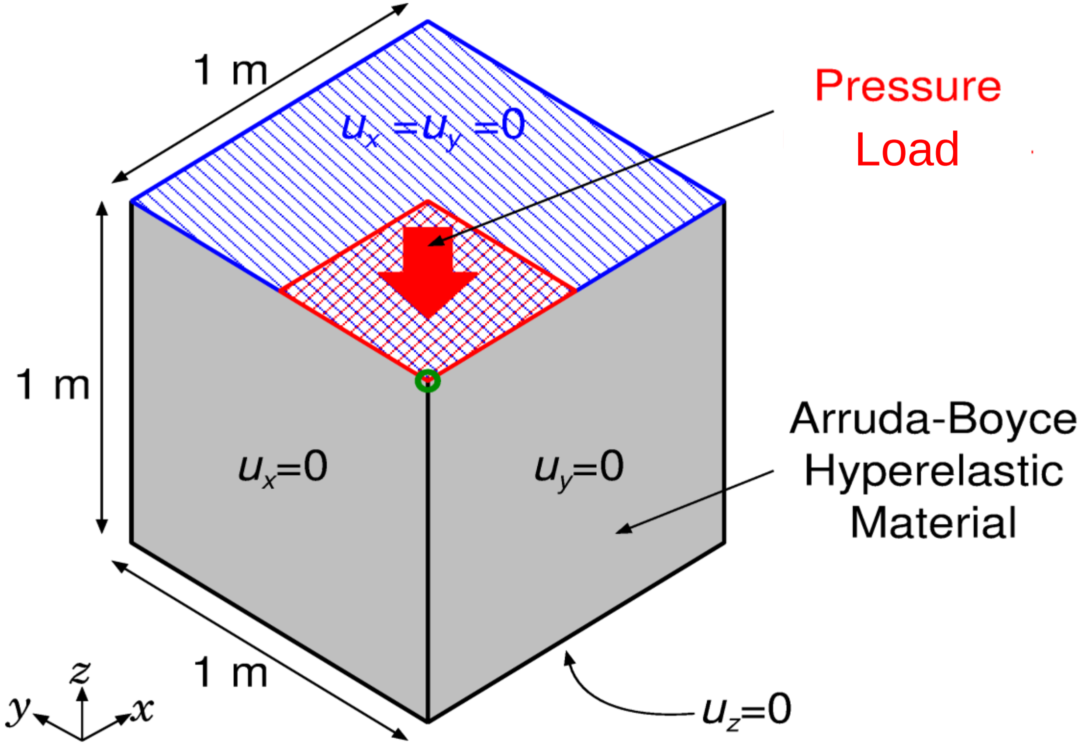
- ✓ Smooth distribution.
- ✓ No locking.

- ➢ Little oscillation, but no checkerboarding.

Static
Implicit

Pressuring of Rubber Block

Outline

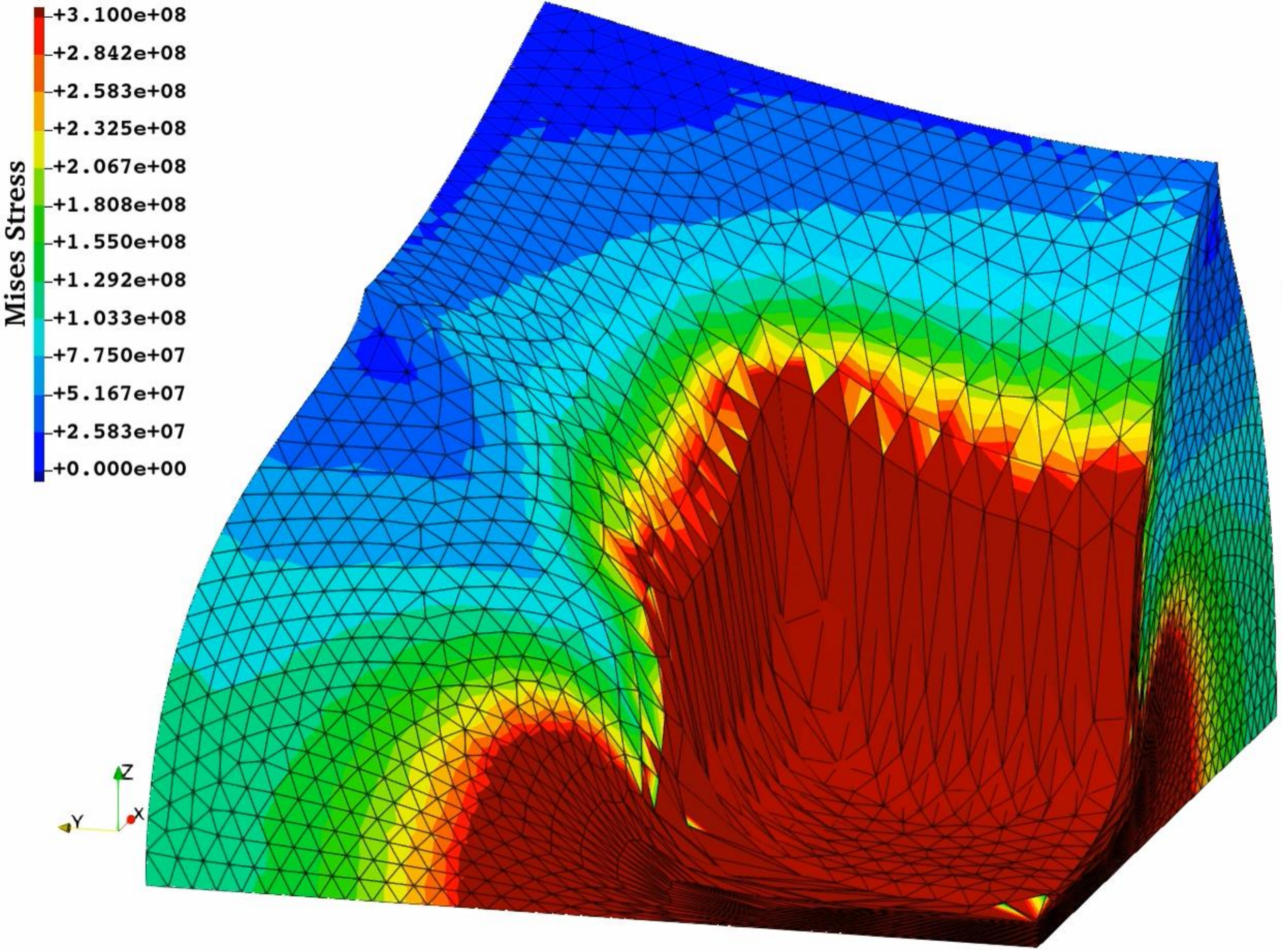


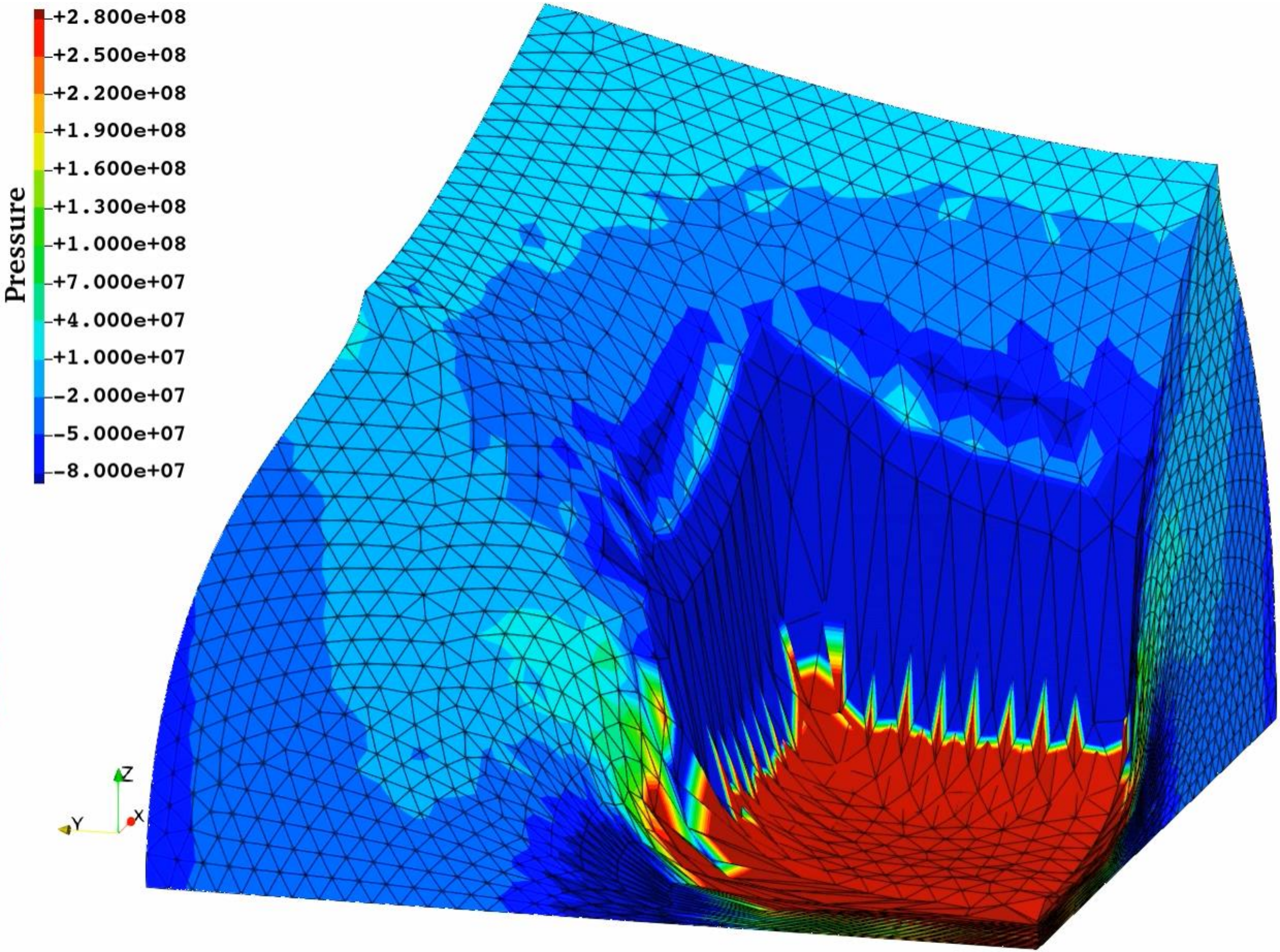
- 1 x 1 x 1 m block.
- Arruda-Boyce hyperelastic material, $E_{\text{ini}} = 24$ GPa, $\boldsymbol{\nu_{\text{ini}} = 0.49}$.
- Applying pressure on ¼ of the top face with lateral confinement.
- Evaluated the result of EC-SSE-SRI-T4.

Static Implicit

Pressuring of Rubber Block

Results of EC-SSE-SRI-T4



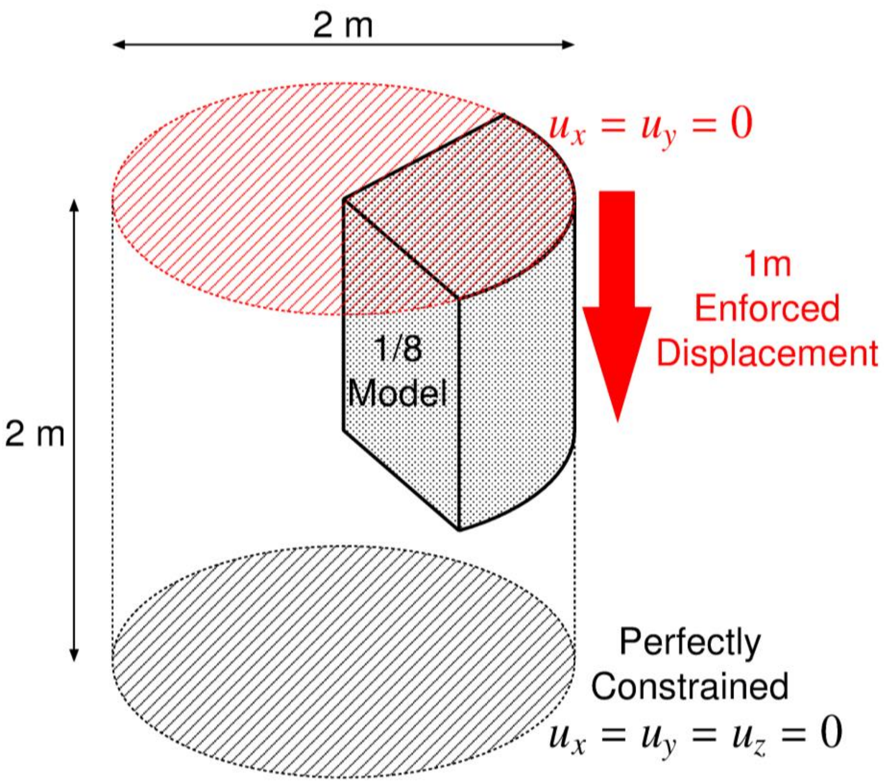


- ✓ No big issue in stress distributions
- ✓ Sufficient large deformation robustness

Static
Implicit

Barreling of Rubber Cylinder

Outline



- 1 m cylinder in radius and height.
- Neo-Hookean hyperelastic material, $E_{\mathrm{ini}} = 6\ \mathrm{GPa}$, $\boldsymbol{\nu}_{\mathbf{ini}} = \mathbf{0.49}$.
- Applying enforced compression displacement on the top face with lateral confinement.
- Evaluated the result of EC-SSE-SRI-T4.

Static Implicit

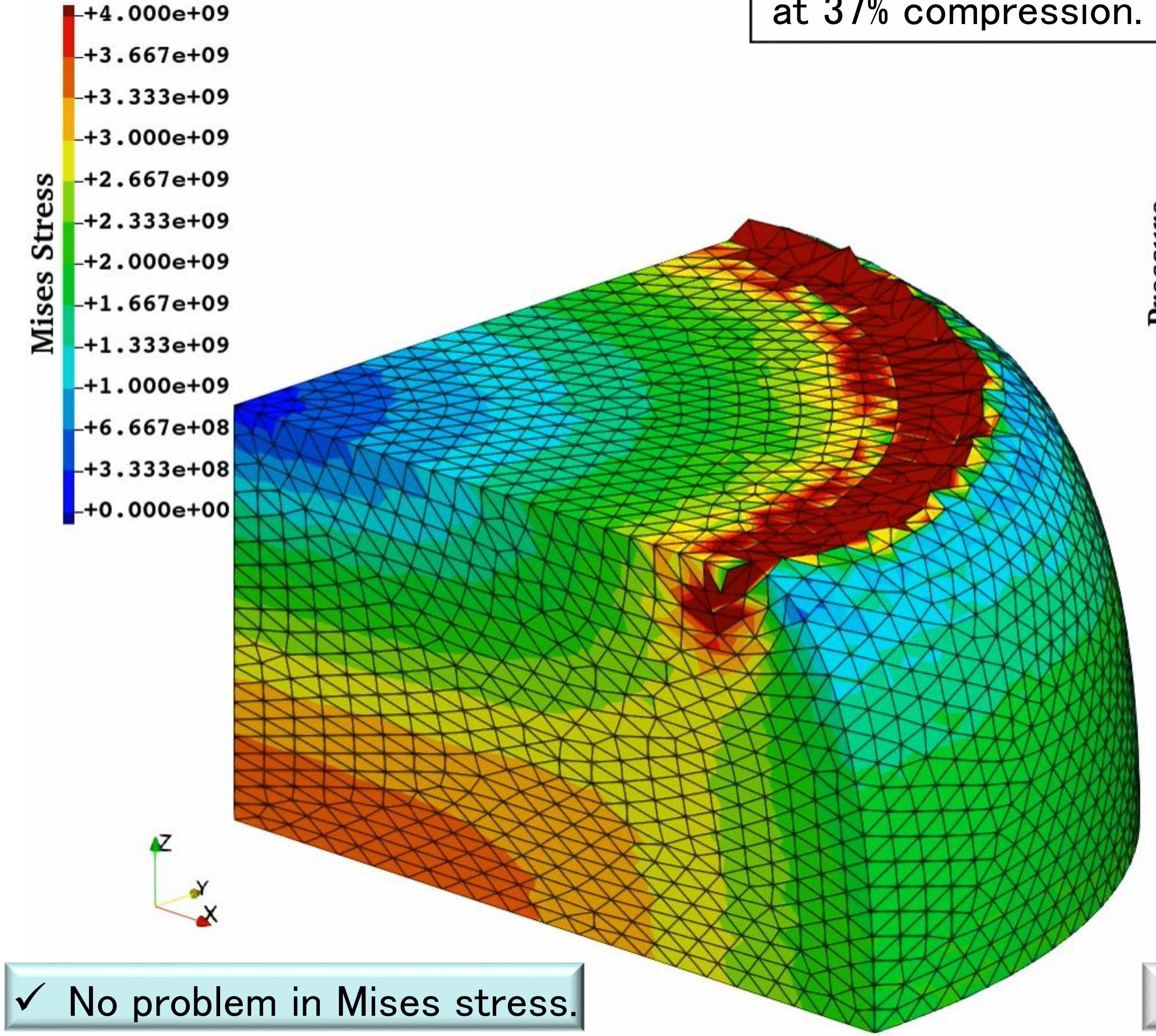
Barreling of Rubber Cylinder

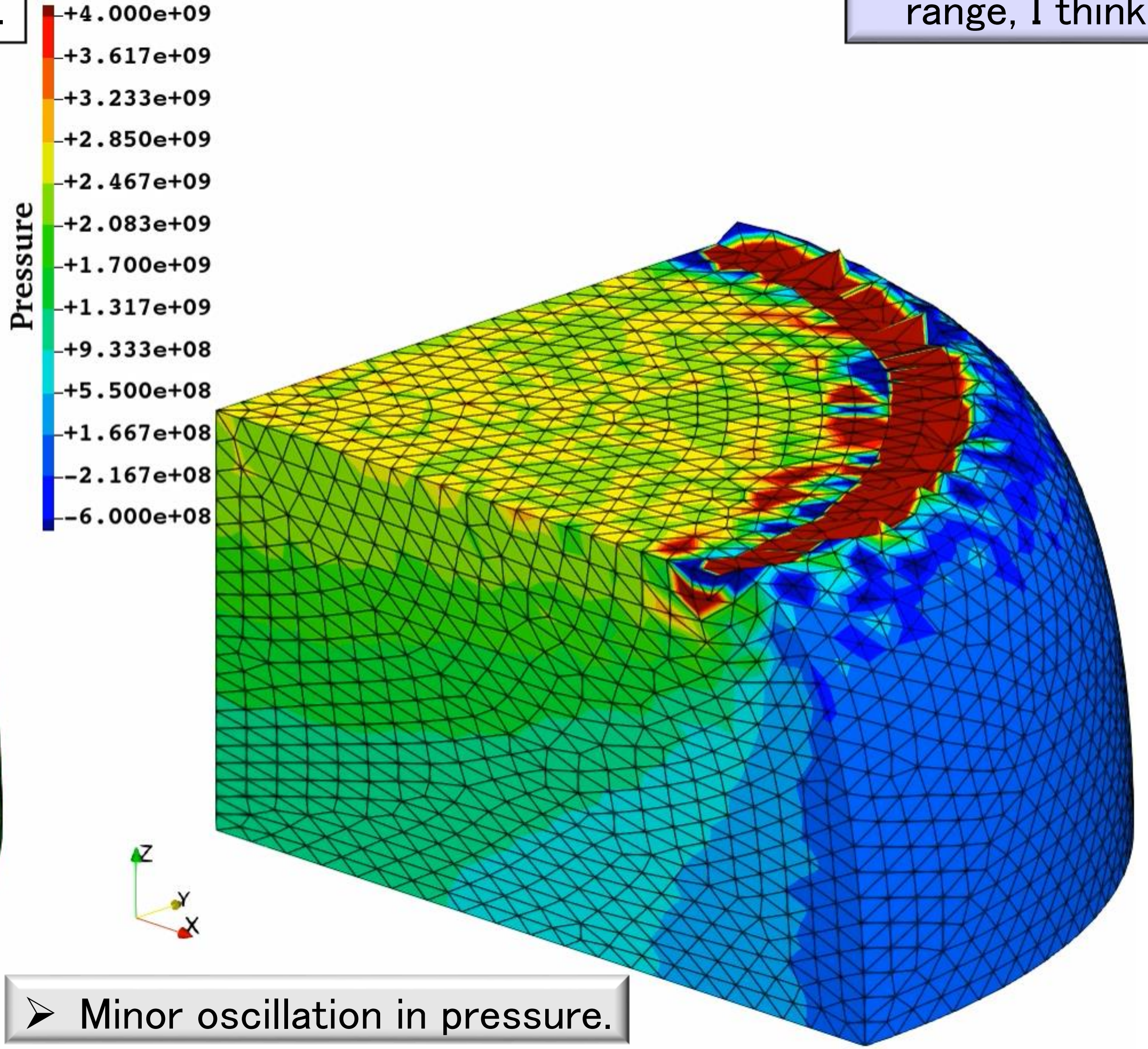
Results of EC-SSE-SRI-T4

Convergence failure at 37% compression.

∴ acceptably robust in large deformation

Within acceptable range, I think.





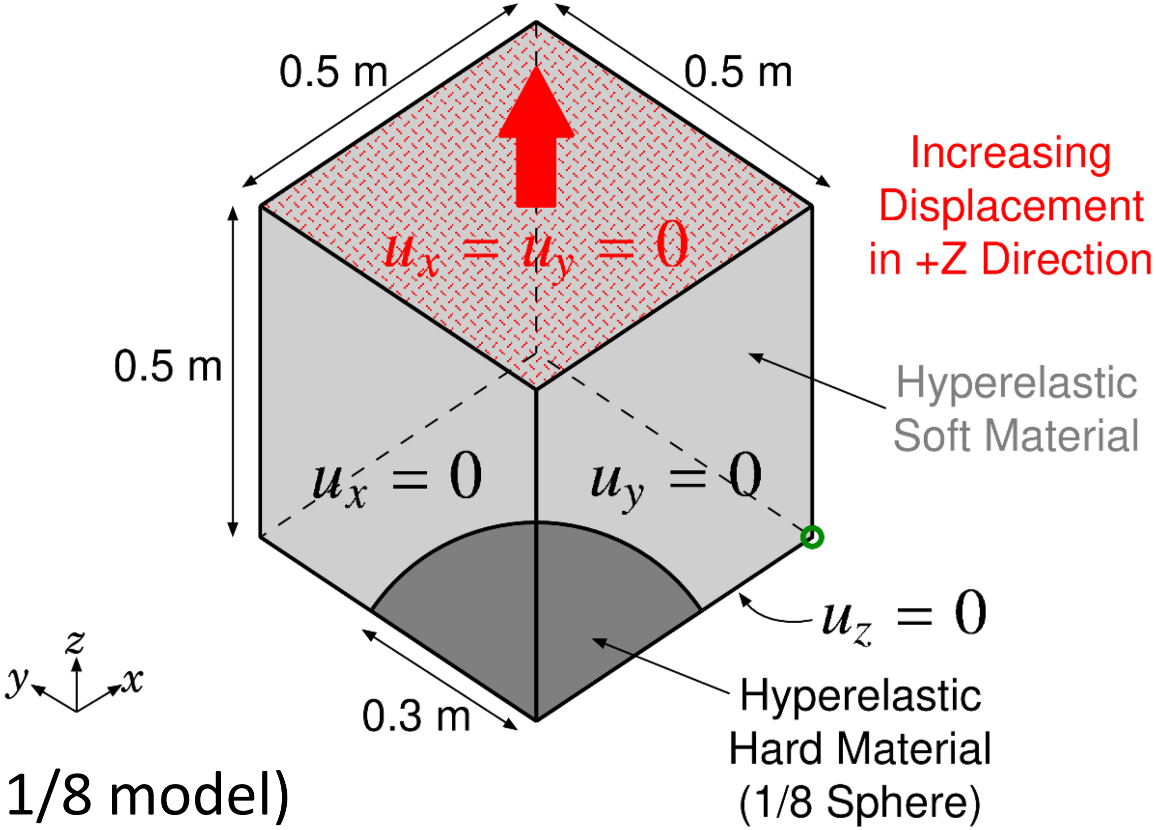
✓ No problem in Mises stress.

➢ Minor oscillation in pressure.

Static
Implicit

Tensioning of Rubber-Filler Composite

Outline



- 0.5 x 0.5 x 0.5 m cube (1/8 model)
- Rubber: Neo-Hookean hyperelastic material ($E_{\mathrm{ini}} = 6\ \mathrm{GPa}, \boldsymbol{\nu}_{\mathbf{ini}} = \mathbf{0.49}$)
- Iron Filler: Neo-Hookean hyperelastic material ($E_{\mathrm{ini}} = 260\ \mathrm{GPa}, \boldsymbol{\nu}_{\mathbf{ini}} = \mathbf{0.3}$)
- Applying enforced tensioning displacement on the top face with lateral confinement.
- Evaluated the result of EC-SSE-SRI-T4.

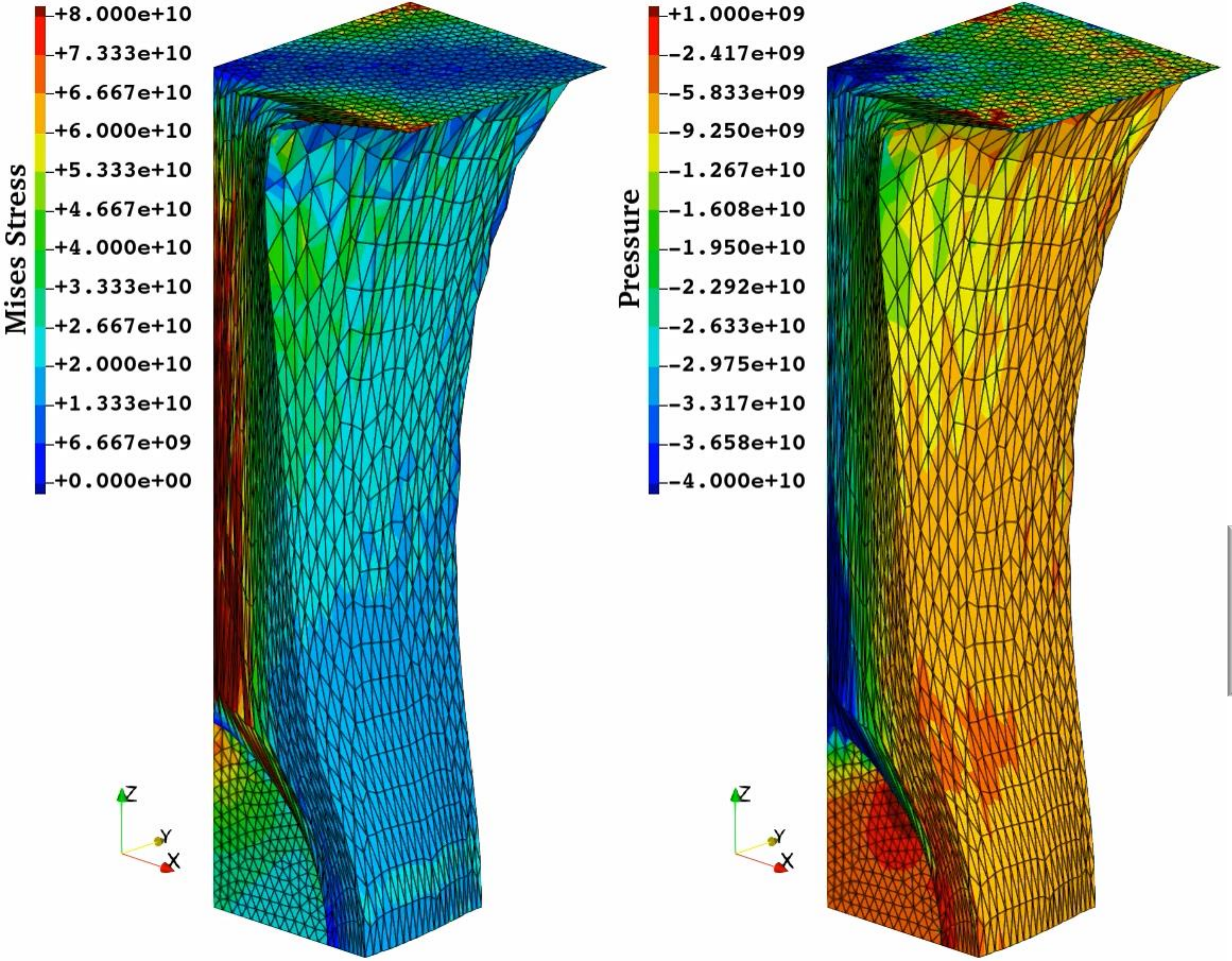
Static
Implicit

Tensioning of Rubber-Filler Composite

Results of
EC-SSE
-SRI-T4

Convergence
failure
at 221% stretch
∴ sufficiently robust
in large deformation

✓ No issue in Mises stress.



➢ Minor pressure oscillation only in rubber part.

Within acceptable range, I think.

Discussion on CPU Time of EC-SSE-SRI-T4

- Since the most of CPU time for implicit analyses is spent solving the stiffness equation (i.e., $[K]\{u\} = \{f\}$), the size of $[K]$ matrix (N) directly affects the CPU time.
- EC-SSE-SRI-T4 is a purely displacement-based FE formulation; thus, the matrix size (N) is exactly identical to that of FEM-T4.
- EC-SSE-SRI-T4 conducts strain smoothing across FE cells; thus, the matrix bandwidth of $[K]$ is x6.7 wider than that of FEM-T4.

Formulation	**Bandwidth of $[K]$**	**v.s. FEM-T4 Ratio**
FEM-T4	14 nodes x 3 DOF	1
FEM-T10	28 nodes x 3DOF	2.0
ES-FEM-T4	45 nodes x 3 DOF	3.2
NS-FEM-T4	60 nodes x 3 DOF	4.3
EC-SSE-T4, EC-SSE-T4-SRI	94 nodes x 3DOF	6.7

- Therefore, as for calculation speed, EC-SSE-SRI-T4 is about x6.7 slower than FEM-T4.

Discussion on CPU Time of EC-SSE-SRI-T4

- Meanwhile, we should remind that
  - FEM-T4 cannot avoid volumetric locking and pressure checkerboarding,
  - FEM-T10 cannot have large deformation robustness (short-lasting),

  no matter how fine the mesh is.
- Therefore, I believe, **EC-SSE-T4-SRI is practically acceptable and worth using**, even though the CPU time is 6.7 times longer than FEM-T4.
  What do you think?

Summary

Summary

- Smoothed finite element methods (S-FEMs) with T4 mesh are quite useful for practical complex geometry problems in various applications, including large deformation analyses.
- EC-SSE-T4 is excellent for compressible solids, and is opening the door of "**S-FEM 2.0**".
- EC-SSE-SRI-T4 is recommended for nearly incompressible solids.
- The EC-SSE family would be the standard T4 formulation in the near future.
- Take home message:

Why not S-FEM?
It is supremely useful and easy to code!

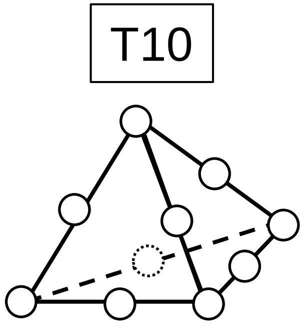
Thank you for your kind attention!

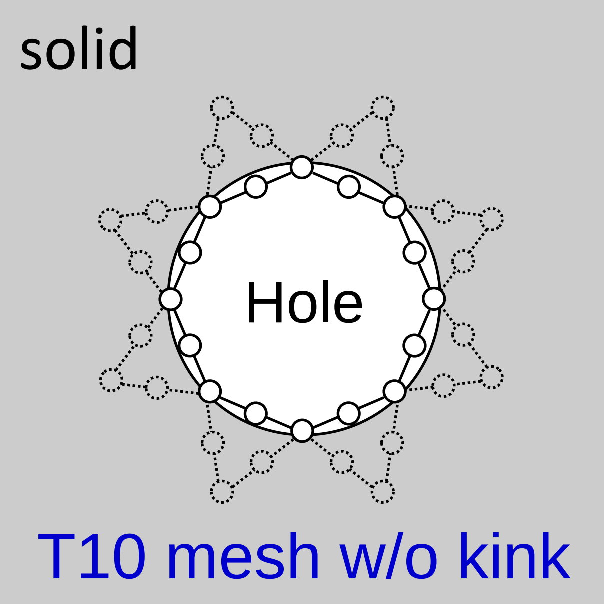
Appendix

Why not T10 but T4?

It is because T10 mesh is NOT good for the representation of complex geometries.

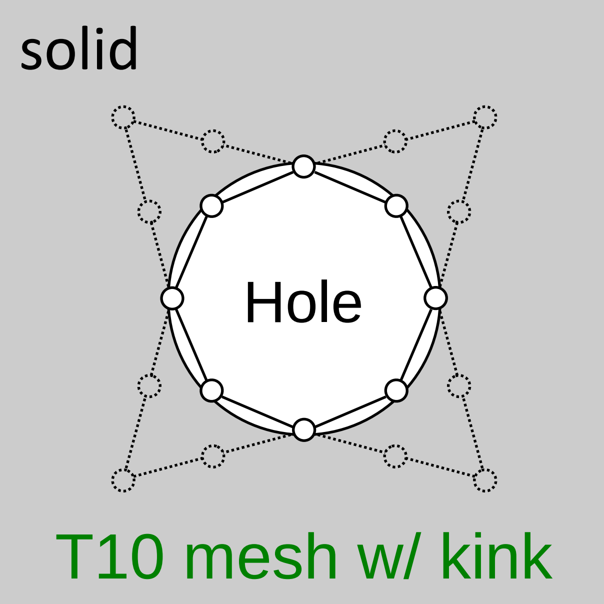
For example, surface mesh around a small hole looks like…





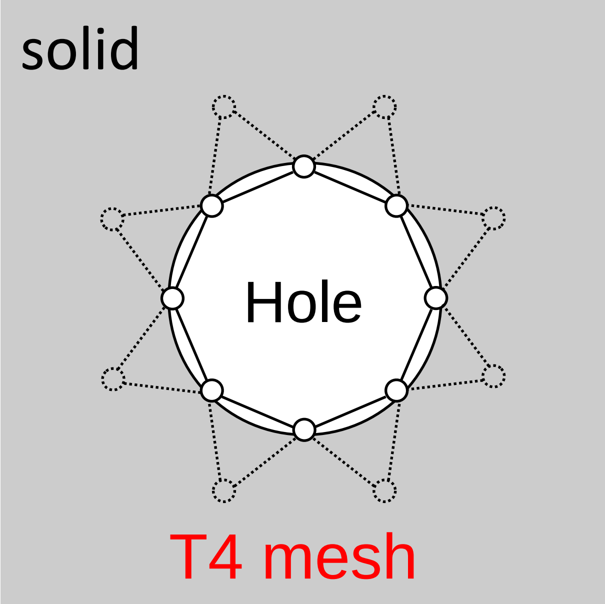
T10 mesh w/o kink

✗ leads to a massive increase in DOF.



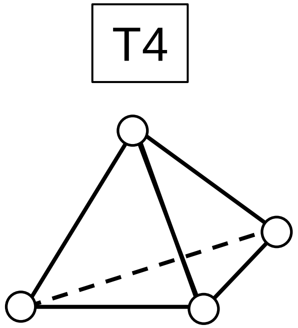
T10 mesh w/ kink

✗ leads to severe accuracy loss.



T4 mesh

✓ is excellent with ES-FEM using minimal # of meshes.



Also, the presence of mid-nodes leads to early convergence failure in large deformation.

Then, T4 is preferable.